# Iterative Methods in Experimental Mathematics

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# Overview

**Goal:** To study the behavior of two different iterative methods which can deal with arbitrarily sized objects

**Method:** Experimental Math!

# Overview

- The 6174 Phenomenon
- The Involution Principle

# Outline

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- Eventually reach the number 6174, at which point forever after:
  - $7641 - 1467 = 6174$

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For any integer $0 \leq \eta \leq b^l - 1$, there is a unique expansion of $\eta$ in base $b$, $\eta = \sum_{i=0}^{l-1} \eta_i \cdot b^i$ $(0 \leq \eta_i \leq b - 1)$.

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- 'Kaprekar function' $K_{l,b} : \{0, 1, ..., b^l - 1\} \to \{0, 1, ..., b^l - 1\}$ :

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- For a fixed $l, b$ define: $P_j = \{K_{l,b}^{(i)}(\eta) \ : \ 0 \le i \le j\}$

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- If haven't stopped before step $i$, iterate again;
  if $K_{l,b}^{(i)}(\eta) \in P_{i-1}$, stop; otherwise, continue.
- At terminating step $t$, we have found $s < t$ s.t. $K_{l,b}^{(t)}(\eta) = K_{l,b}^{(s)}(\eta)$.
  The list of intermediate values is the *limiting orbit* of $\eta$ under the Kaprekar routine:

$$O_{l,b}(\eta) = \left[K_{l,b}^{(s)}(\eta)\ ,\ K_{l,b}^{(s+1)}(\eta)\ ,\ \dots\ ,\ K_{l,b}^{(t-1)}(\eta)\right]$$

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Questions:

Is it *always* true for every fixed $l$ that there exists some number $s$ such that $O_{l,10}(\eta) = [s]$ for every non-degenerate $\eta$ in the domain?

More generally, is it *always* true for every fixed $b$ and $l$ that there exists some number $s$ such that $O_{l,b}(\eta) = [s]$ for every non-degenerate $\eta$ in the domain?

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  - $O_{2,10}(\eta) = [09, 81, 63, 27, 45]$
  - $O_{7,10}(\eta) = [7509843, 9529641, 8719722, 8649432, 7519743, 8429652, 7619733, 8439552]$

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  - For each fixed $l$, is there an orbit $S = [s_1, s_2, ..., S_N]$ such that $O_{l,10}(\eta) = S$ for all $\eta$?

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- Still no, fails to be a single limiting orbit for $l = 5, 6, 8$, among many other cases
- e.g. $O_{5,10}(85932) = [62964,\ 71973,\ 83952,\ 74943]$ while
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So, what *can* we say?

# More Notation

- Can express $\eta = \sum_{i=0}^{l-1} \eta_i \cdot b^i$ by the list $[\eta_{l-1}, \eta_{l-2}, ..., \eta_2, \eta_1, \eta_0]$; when base is understood such a list uniquely determines the integer $\eta$

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  - e.g. $Orb(5, 6) = \{[\ [4, 1, 5, 3, 2]\ ]\ ,\ [\ [3, 1, 5, 3, 3], [3, 5, 5, 5, 2]\ ]\}$

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- $N(l, b) = |Orb(l, b)|$
- Enumerating the set $Orb(l, b)$ as a list $[o_1, o_2, ...o_{N(l,b)}]$ in order of non-decreasing orbit size :
  $L(l, b) = [|o_1|, |o_2|, ..., |o_{N(l,b)}|]$
  - e.g. $L(5, 6) = [1, 2]$

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**Theorem (Even Base)**

*If $b = 2n$ is even, then $N(3, b) = 1$ and $L(3, b) = [1]$. In particular,*

$$Orb(3, b) = [\,[n-1, 2n-1, n]\,]$$

**Theorem (Odd Base)**

*If $b = 2n+1$ is odd, then $N(3, b) = 1$ and $L(3, b) = [2]$. In particular,*

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# A Simple Proposition

**Proposition**

*Suppose* $\eta = [\eta_0, \eta_1, \eta_2]$*, where* $\eta$ *is non-degenerate, and Let* $\delta = \delta(\eta)$ *be the difference between the largest and smallest* $\eta_i$ *(*$\delta \geq 1$*). Then for any base b, the successor* $K(\eta)$ *to* $\eta$ *under one iteration of the Kaprekar function is:*

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*Suppose $\eta = [\eta_0, \eta_1, \eta_2]$, where $\eta$ is non-degenerate, and Let $\delta = \delta(\eta)$ be the difference between the largest and smallest $\eta_i$ ($\delta \geq 1$). Then for any base b, the successor $K(\eta)$ to $\eta$ under one iteration of the Kaprekar function is:*

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## Proof.

WLOG $\eta_2 \geq \eta_1 \geq \eta_0$, so $\delta = \eta_2 - \eta_0$:

$$\begin{array}{rrlllll} & M(\eta^*) = & \eta_2 \cdot b^2 & + & \eta_1 \cdot b & + & \eta_0 \\ - & m(\eta^*) = & \eta_0 \cdot b^2 & + & \eta_1 \cdot b & + & \eta_2 \\ \hline & K(\eta^*) = & \delta \cdot b^2 & + & & - & \delta \\ & = & (\delta - 1) \cdot b^2 & + & (b - 1) \cdot b & + & b - \delta \end{array}$$

Thus $K(\eta) = [\delta - 1, b - 1, b - \delta]$ □

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We have just shown the successor $K(\eta)$ to $\eta$ under one iteration of the Kaprekar function is: $K(\eta) = [\delta - 1, b - 1, b - \delta]$

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5. Then at the next step $x$ will decrease by 1 and $y$ will increase by 1, and their sum remains $b - 1$
6. This continues until we reach a number where $x < y$, which first happens when $x = \lfloor b/2 \rfloor$.

# A Result for $l = 4$

Hasse and Prichett (1978):[1]

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**Theorem**

*If* $b = 5 \cdot 2^n$ *for* $n \geq 1$, *then:*

- $[\,[3 \cdot 2^n, 2^n - 1, 2^{n+2} - 1, 2^{n+1}]\,] \in Orb(4, b)$
- *If n is odd,* $N(4, b) = 1$ *and* $L(4, b) = [1]$.
- *If n is even,* $N(4, b) = 2$ *and*
    - *If* $n \pmod 4 = 0$, $L(4, b) = [1, n + 1]$, *and the length* $n + 1$ *orbit is generated by* $[2^n - 1, 5 \cdot 2^n - 1, 5 \cdot 2^n - 1, 2^{n+2}]$
    - *if* $n \pmod 4 = 2$ *then* $L(4, b) = [1, 2(n + 1)]$, *and the length* $2(n + 1)$ *orbit is generated by* $[2^{n+1} - 1, 5 \cdot 2^n - 1, 5 \cdot 2^n - 1, 3 \cdot 2^n]$

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Note: Essentially entire paper was to prove this theorem

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# How to find and prove patterns in general?

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- Problem: exponential growth
- We need (computer) help, e.g. Maple

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- E.G. `listPhenom([1, 9, 8, 9], 10) =`

$$[\,4, 1, [\,[6, 1, 7, 4]\,], [\,[1, 9, 8, 9], [8, 0, 8, 2], [8, 5, 3, 2], [6, 1, 7, 4], [6, 1, 7, 4]\,]\,]$$

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- Can improve "a little" by recalling order of entries irrelevant (e.g. don't need to look both at $[2,4,3,1]$ and $[3,1,2,4]$)
    - Number of distinct lists: $\binom{b+l-1}{l}$ (e.g. correspond to distinct monomials in $b$ variables with total degree $l$)
    - e.g. for $l = 4$, $b = 2^4$, $(2^4)^4 - 2^4 = 65520$ vs. $\binom{2^4+3}{4} = 3876$ integers
    - BUT for fixed $l$: $\binom{b+l-1}{l} \sim \frac{(b+l-1)^l}{l!} = \Omega(b^l) = \Omega(K^n)$

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`getNPCconjecturesPaper`: Demo

# My Conjectures

## Length 4

### Conjecture (Base $2^n$)

*If $b \geq 4$ is a positive integer of the form $b = 2^n$, then:*

1. *If $n$ is even, $N(4, b) = n$. Specifically, writing $n = 2k$ we have that*

$$L(4, b) = [k, k+1, 2k, ..., 2k, 2(k+1), ...2(k+1)].$$

*where the number of $2k$'s and $2(k+1)$'s is $k-1$.*

2. *If $n$ is odd, $N(4, b) = n - 1$. Specifically, writing $n = 2k + 1$ we have that*

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# My Conjectures

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**Conjecture (Base $3 \cdot 2^n$)**

*If $b \geq 6$ is a positive integer of the form $b = 3 \cdot 2^n$, then:*

1. *If $n$ is even, $N(4, b) = 2$. Specifically, writing $n = 2k$ we have that $L(4, b) = [2k+1, 2(2k+1)]$. In particular, the orbit of length $2k+1$ is generated by $[2^n - 1, 3 \cdot 2^n - 1, 3 \cdot 2^n - 1, 2^{n+1}]$ and the orbit of length $2(2k+1)$ is generated by $[2^n, 2^{n-1} - 1, 5 \cdot 2^{n-1} - 1, 2^{n+1}]$.*
2. *If $n$ is odd, $N(4, b) = 1$. Specifically, writing $n = 2k+1$ we have that $L(4, b) = [6(k+1)]$. In particular, the singular orbit is generated by $[2^n, 2^{n-1} - 1, 5 \cdot 2^{n-1} - 1, 2^{n+1}]$.*

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**Conjecture (Base $7 \cdot 2^n$)**

*If $b \geq 14$ is a positive integer of the form $b = 7 \cdot 2^n$, then:*

1. *If $n \pmod 3 = 0$, $N(4, b) = 2$. Specifically, writing $n = 3k$ we have that $L(4, b) = [3, 3k+1]$. In particular, orbit of length 3 is generated by $[3 \cdot 2^n, 2^n - 1, 3 \cdot 2^{n+1} - 1, 2^{n+2}]$ and the orbit of length $3k+1$ is generated by*
$$[2^n - 1, 7 \cdot 2^n - 1, 7 \cdot 2^n - 1, 3 \cdot 2^{n+1}]$$
2. *If $n \pmod 3 \neq 0$, $N(4, b) = 1$ and $L(4, b) = [3]$. In particular, the singular orbit is generated by:*
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**Conjecture (Base $3^n$)**

*If $b \geq 9$ is a positive integer of the form $b = 3^n$, then $N(4, b) = \sum_{i=1}^{n-1} 3^i - 1 = \frac{3^n - 3}{2} - (n - 1)$. In particular $L(4, b) = [3, 3, ..., 3^k, ..., 3^k, ..., 3^{n-1}, ..., 3^{n-1}]$, where $k$ ranges from 1 to $n - 1$ and the integer $3^k$ appears $3^k - 1$ times in $L(4, b)$*

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2. Orbit Lengths Division Conjecture: *Let $\mu = \max\{l : l \in L(4, b)\}$, i.e. $\mu$ is the maximum length of a limiting orbit. Then each integer in $L(4, b)$ divides $\mu$.*
3. Elements Conjecture: *Let $U$ be the set of all 4-digit numbers which occur in some limiting orbit for base b and length 4, then:*

$$U = \bigcup_{k=1}^{n-1} \bigcup_{i=0}^{k-1} \{ \, [\, 2k+1, 2i, 2(n-i)-1, 2(n-k) \,] \, \}$$

*In particular*[a], $|U| = \binom{n}{2}$.

[a] In the 6174phenom.txt file, the procedure correspondence(n,k,m) gives the bijective counterpart in the set $U$ of the 2-subset {k,m} of [n].

# My Conjectures

## Length 7

**Conjecture (Base 0 (mod 4))**

*If b is an integer of the form $b = 4n$ for $n \geq 6$, then $N(7, b) = 2$ and $L(7, b) = [1, 19]$ In particular, the length 1 orbit is given by:*

$$[\ [3n, 2n, n-1, 4n-1, 3n-1, 2n-1, n]\ ]$$

*and the length 19 orbit is generated by:*

$$[3n-1, 2n-2, n-2, 4n-1, 3n, 2n+1, n+1]$$

**Conjecture (Base 2 (mod 4))**

*If $b = 4n + 2$ is an integer of the form $b = 4n + 2$ for $n \geq 4$, then $N(7, b) = 1$ and $L(7, b) = [4]$. In particular, the single limiting orbit is given by:*

$$[[3n+1, 2n, n-2, 4n+1, 3n+2, 2n+1, n+1], [3n+3, 2n+1, n, 4n+1, 3n, 2n, n-1],$$

$$[3n+2, 2n+3, n-1, 4n+1, 3n+1, 2n-2, n], [3n+2, 2n+2, n+2, 4n+1, 3n-2, 2n-1, n]]$$

# Outline

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- Have bijections $f : A \to B$ and $g : A' \to B'$.
- Want to find a bijection $h : \overline{A'} \to \overline{B'}$

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- Note a mathematician only visits the seat originally assigned to him and seats assigned to physicists, and so in particular he cannot end up in a seat originally assigned to another mathematician (and cannot end up stuck in loop e.g. $A \to B \to C \to A$)

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- An iteration of our method: A mathematician $a$ checks seat $b$. If $b$ not occupied by a physicist, then $a$ sits in $b$ ($h(a) = b$). Otherwise arrives at next seat by $f \circ g^{-1}(b)$

# Proof Ideas

For each $a \in \overline{A'}$, find $h(a)$ in $\leq k+1$ iterations:

- Recall our mathematician cannot get stuck in a loop; in particular must end up at a new seat at each iteration
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  - Say $a$ reaches $b$ in $s+1$ iterations, $\hat{a}$ in $t+1$

$$h(a) = (f \circ g^{-1})^{(s)} \circ f(a) \quad \text{and} \quad h(\hat{a}) = (f \circ g^{-1})^{(t)} \circ f(\hat{a})$$

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$$h(a) = (f \circ g^{-1})^{(s)} \circ f(a) \quad \text{and} \quad h(\hat{a}) = (f \circ g^{-1})^{(t)} \circ f(\hat{a})$$

  - $(f \circ g^{-1})^{(s-t)} \circ f(a) = f(\hat{a})$
    - Case I: $s - t = 0 \quad \rightarrow \quad$ Contradiction.
    - Case 2: $s - t \geq 1 \quad \rightarrow \quad$ Contradiction.

# InvolutionPrinciple.txt

## `compBijection`

Want to analyze behavior for large $n = |A|$.

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- Let $A = \{a_1, a_2, ..., a_n\}$ enumerate $A$, $b_i = f(a_i)$ enumeration of $B$
- $A' = \{a_i \ : \ i \in I_{A'}\}$, and $B' = \{b_i \ : \ i \in I_{B'}\}$ for some index sets $I_{A'} \subset [n]$ and $I_{B'} \subset [n]$ of size $k = |A'| = |B'|$.

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- Let $A = \{a_1, a_2, ..., a_n\}$ enumerate $A$, $b_i = f(a_i)$ enumeration of $B$
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# InvolutionPrinciple.txt

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- e.g. triple $indA' = [4, 1, 5]$, $indB' = [6, 3, 4]$, $\tau = [3, 1, 2]$ says:
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## `fullCompBijection` and `fullCompBijectionComplexity`

To look at full bijection, use `fullCompBijection`$(n, indA', indB', \tau)$, which outputs all pairs $[a_i, h(a_i)]$ for $a_i \in \overline{A'}$, e.g:

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- For example, for $h$ constructed above: $c(a_2) = 1, c(a_3) = 2, c(a_6) = 2$, so $C(h) = c(a_2) + c(a_3) + c(a_6) = 1 + 2 + 2 = 5$; and $\overline{C(h)} = \frac{5}{3}$

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- $[C(h), \overline{C(h)}] = \texttt{fullCompBijectionComplexity}(n, indA', indB', \tau)$

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- Should have:

$$\texttt{runTest}(n, k, N) \to \mathbb{E}[C(h)],$$

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- $\Pr[c(a) = m] = \left( \prod_{i=1}^{m-1} \frac{k-(i-1)}{n-(i-1)} \right) \cdot \frac{n-k}{n-(m-1)}$

$$= (n-k) \cdot \left( \prod_{j=0}^{m-1} \frac{1}{n-j} \right) \cdot \left( \prod_{j=0}^{m-2} (k-j) \right) = (n-k) \cdot \frac{1}{(n)_m} \cdot (k)_{m-1}$$

$$= (n-k) \cdot \frac{(m-1)! \cdot \binom{k}{m-1}}{m! \binom{n}{m}} = \frac{(n-k)}{m} \cdot \frac{\binom{k}{m-1}}{\binom{n}{m}}$$

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# Analyzing How Complexity changes with n

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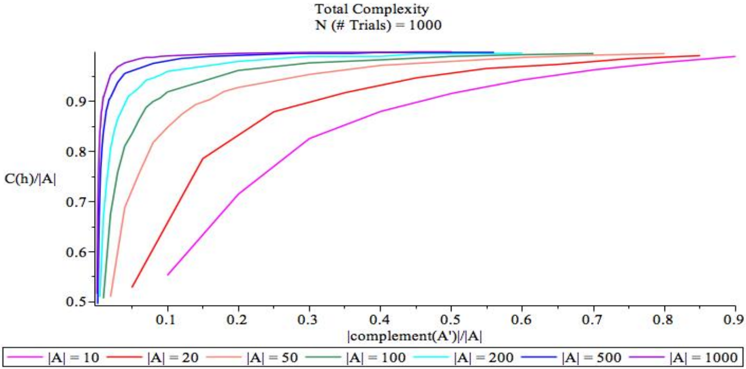


Figure: Complexity Analysis for variable $n$ between 10 and 1000

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- Analogous procedures in 6174phenom.txt which study number of iterations it takes to reach limiting orbit

# The End

Questions?