Flexible Schemes and Beyond: Experimental Enumeration of Pattern-Avoiding Permutations

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Intro to Pattern Avoidance

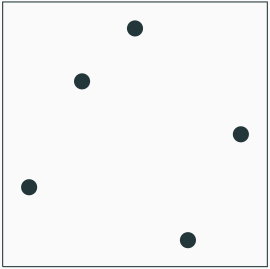
Permutations

Definition

A permutation is a reordering of the numbers in $[n] = \{1, 2, \ldots, n\}$.

Example

24513 is a permutation (of $[5]$*)*



Pattern Containment

Definition

A permutation $\pi = \pi_1\pi_2\ldots\pi_n$ *contains a pattern* $\sigma = \sigma_1\sigma_2\ldots\sigma_k$ *if there exists an increasing sequence* $i_1, i_2, \ldots, i_k$ *such that* $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$ *has the same relative order as* σ*.*

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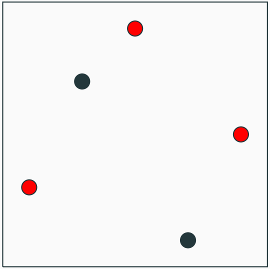
Example

$\underline{2}4\underline{5}1\underline{3}$ contains 132.

Example

24513 does not contain 321.



Pattern Avoidance

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If π fails to contain σ, then π avoids σ.

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$\mathrm{Av}(\sigma; n) = \{\pi \in S_n : \pi \text{ avoids } \sigma\}$.

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$\mathrm{Av}(12; 3) = \{321\}$

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Question

For fixed σ, find a formula/generating function/recurrence for $|\operatorname{Av}(\sigma; n)|$.

Classical Results

Theorem (MacMahon, 1915)

$|\operatorname{Av}(123; n)| = C_n = \frac{1}{n+1}\binom{2n}{n}$

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$|\operatorname{Av}(12\dots k, l(l-1)\dots 1; n)| = 0$ *for* $n \geq (k-1)(l-1)+1$

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Theorem (Knuth, 1968)

$|\operatorname{Av}(\sigma; n)| = C_n$ *for each* $\sigma \in S_3$

Enumeration Schemes

Overview

- One of several automatic enumeration techniques
- Fully rigorous
- References:
    - Zeilberger '98
    - Vatter '05
    - Pudwell '08
    - Baxter and Pudwell '12

Why Experimental Math?

- One brain
- Many processors

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- One brain
- Many processors
- Humans make mistakes
- Computers don't
- Processors always improving
- No human upgrades in 200,000 years (despite bugs)

General Goal

- Input a pattern or set of patterns
- Produce a certificate
- Use certificate to find $|\text{Av}(\sigma; n)|$ in poly(n) time

Key Object of The Recurrence

Definition

$$Z(\sigma, d_1 \dots d_k, [g_1, \dots, g_{k+1}]) = \Big\{\pi \in S_{k+\sum g_i} : \pi_{g_1+1} = d_1, \pi_{g_1+g_2+2} = d_2, \dots, \pi_{g_1+g_2+\dots+g_k+k} = d_k\Big\}$$

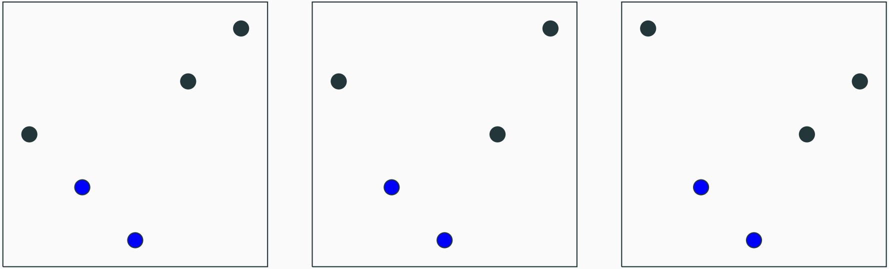
Definition

$d_1 \dots d_k$ is the downfix

An Example

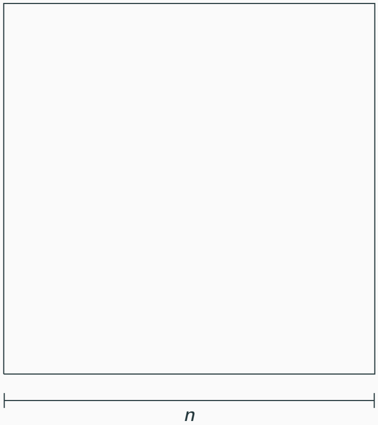
$$Z(132, 21, [1,0,2]) = \{32145, 42135, 52134\}$$

32145 42135 52134



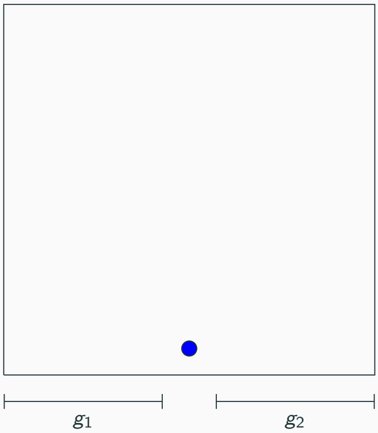
Building a Permutation

Our Goal: Find $|\operatorname{Av}(123; n)| = |Z(123, \emptyset, [n])|$.



Building a Permutation

$$Z(123, 1, [g_1, g_2])$$



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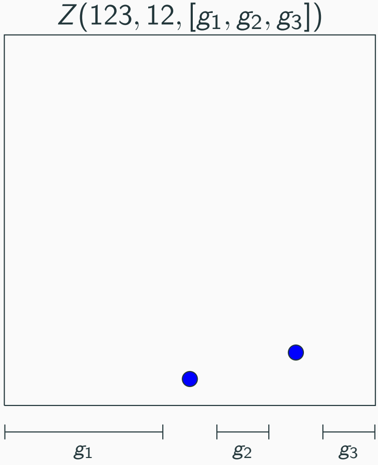
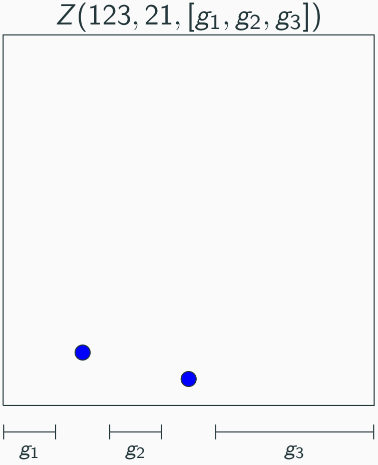
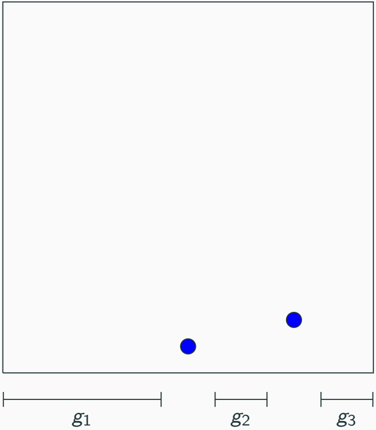


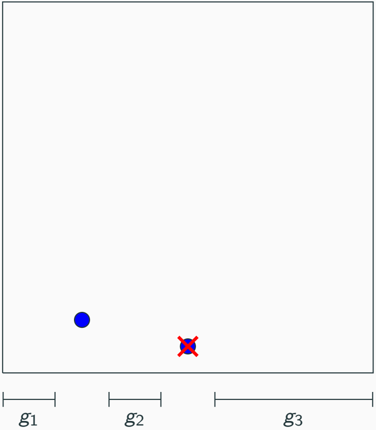


Building a Permutation

$Z(123, 12, [g_1, g_2, g_3])$

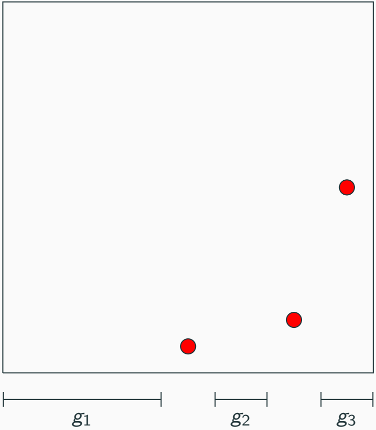


$Z(123, 1, [g_1, g_2 + g_3])$

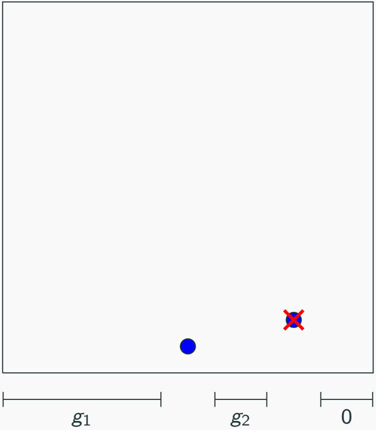


Building a Permutation

If $g_3 > 1 : \{\}$



If $g_3 = 0 : Z(123, 1, [g_1, g_2])$



The Resulting Recurrence

$$|Z(123, \emptyset, [n])| = \sum_{i=0}^{n-1} |Z(123, 1, [i, n-i-1])|$$

$$|Z(123, 1, [g_1, g_2])| = \sum_{j=0}^{g_1-1} |Z(123, 1, [j, g_1+g_2-j-1])| + \sum_{j=0}^{g_2-1} |Z(123, 12, [g_1, j, g_2-j-1])|$$

$$|Z(123, 12, [g_1, g_2, g_3])| = \begin{cases} |Z(123, 1, [g_1, g_2]) & \text{if } g_3 = 0 \\ 0 & \text{if } g_3 > 0 \end{cases}$$

Algorithm

1. Input downfix $d_1 d_2 \ldots d_k$ and pattern σ
2. Determine which *gap vectors* $[g_1, \ldots, g_{k+1}]$ ensure σ appears ($g_3 > 0$ in the last example).
3. If $[g_1, \ldots, g_{k+1}]$ is not such a gap vectors, can any d_i be deleted (2 in the last example)?
4. If so, rewrite $|Z(\sigma, d_1 d_2 \ldots d_k, [g_1, \ldots, g_{k+1}])|$ using a shorter downfix. If not, separate $Z(123, d_1 d_2 \ldots d_k, [g_1, \ldots, g_{k+1}])$ into cases by where next element occurs.

Insertion Encoding

- Technique for constructing a permutation element by element
- References:
    - Albert, Linton, and Ruškuc '05
    - Vatter '12

Map Insertions to Letters

$\diamond$	
$\diamond 1 \diamond$	m_1
$2 \diamond 1 \diamond$	$m_1 l_1$
$2 \diamond 13$	$m_1 l_1 f_2$
$24 \diamond 13$	$m_1 l_1 f_2 l_1$
24513	$m_1 l_1 f_2 l_1 f_1$

Regular Encodings

- Consider the strings corresponding to permutations in $\mathrm{Av}(\sigma; n)$ (the "language")
- If this language is *regular*, then there is a fast (linear-time) algorithm to calculate $|\mathrm{Av}(\sigma; n)|$.

Theorem (Albert, Linton, and Ruškuc)
A permutation class has a regular insertion encoding iff it contains finitely many vertical alternations.

Flexible Schemes: Goal and Main Idea

- Generalize enumeration schemes so that they count every permutation class with regular insertion encoding

Flexible Schemes: Goal and Main Idea

- Generalize enumeration schemes so that they count every permutation class with regular insertion encoding
- What if there are restrictions on $\mathbf{g}$ that don't make $|Z(\sigma, \pi, \mathbf{g})| = 0$, but do let us rewrite it as $|Z(\sigma, \pi', \mathbf{g}')|$?

A (Partial) Example I

- Avoiding 1423, 2314
- Try to reduce the downfix 321
- $Z(\{1423, 2314\}, 321, [g_1, g_2, g_3, g_4])$
- Suppose $g_2 \geq 1$, we can delete 1 in the downfix.
- Suppose $g_2 = 0$, we can delete 3 in the downfix.

Schemes to a Computer

- A scheme is a set of replacement rules
- E.g. $\Big[[3,2,1],\Big[\big[[0,1,0,0],3\big],\big[[0,0,0,0],1\big]\Big]\Big]$ means that a 321 downfix can have the third element deleted if $g_2 \geq 1$, and otherwise can have the first element be deleted.

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- For 123-avoiders, we have the scheme

$$\Big\{\Big[[1,2],\Big[\big[[0,0,1],0\big],\big[[0,0,0],2\big]\Big]\Big],$$
$$\Big[[2,1],\Big[\big[[0,0,0],2\big]\Big]\Big]\Big\}$$

- To compute $|Z(123,\emptyset,[n])|$, consider all possible length two downfixes, then reduce with replacement rules

Theorem

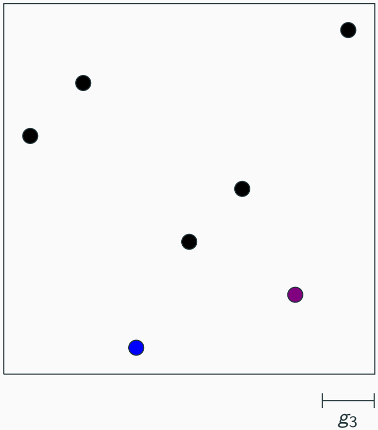
Let B be a set of forbidden patterns, π be a downfix, and $\mathbf{h}, \mathbf{h_1}, \mathbf{h_2}, \ldots, \mathbf{h_k}$ be gap conditions. Suppose that

$$|Z(B, \pi, \mathbf{g})| = |Z(B, d_r(\pi), d_r(\mathbf{g}))|$$

for all $\mathbf{g}$ with $||\mathbf{g}||_1 \leq ||B||_\infty - 1 + ||\mathbf{h}||_1$ which satisfy $\mathbf{h}$ but fail to satisfy any of $\mathbf{h_1}, \mathbf{h_2}, \ldots, \mathbf{h_k}$. Then the equality holds for all $\mathbf{g}$ which satisfy $\mathbf{h}$ but fail to satisfy any of $\mathbf{h_1}, \mathbf{h_2}, \ldots, \mathbf{h_k}$.

Proof By Picture

- $\mathbf{h} = [0, 0, 1], \pi = 12, B = \{321\}, \mathbf{g} = [2, 2, 1]$
- Suppose there is some σ which would be in $Z(B; \pi; \mathbf{g})$, except it contains a forbidden pattern and deleting the r^{th} element of the downfix removes this pattern



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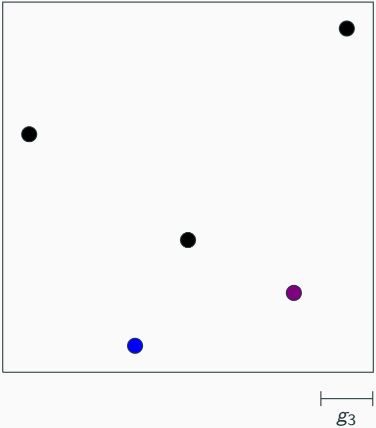


Regular Insertions Encodings Imply Flexible Schemes

Theorem

Let B be a set of forbidden patterns, and suppose the class of permutations avoiding B has a regular insertion encoding. Then, that class also has a finite flexible scheme.

Proof Outline

- Recall that Regular Insertion Encoding $\iff$ finitely many vertical alternations (say all have length $\leq 2k$)
- Any gap vector has $\leq k$ positive entries or gives no valid permutations
- Choose some downfix π and gap vector $\mathbf{g}$ with $\leq k$ positive entries
- For a downfix element i not to be deletable, there must be some permutation which would be in $Z(B, \pi, \mathbf{g})$ except there is a forbidden pattern which uses i.
- Each σ can only use finitely many i, and there are only finitely many possible σ; so if π is long enough there is a deletable element.

Empirical Results

Table 1: Empirical Results

Pat length	Sym Classes	Ins. Enc.	ES	FS	New with FS
[3]	2	0	2	2	0
[4]	7	0	2	2	0
[5]	23	0	2	2	0
[3], [3]	5	5	5	5	0
[4], [4]	56	13	33	44	9
[4], [5]	434	30	112	173	59

Drawbacks of Flexible Schemes

- May require long gap conditions
- Give $O(n^{d+2})$ time enumeration where d is the depth of the scheme.

Covincular Pattern Avoidance

Definition

A *covincular* pattern (σ, X) consists of a permutation σ and a set $X \subseteq [|\sigma| - 1]$.

Definition

A permutation π *contains* (σ, X) if there exists a sequence $i_1, i_2, \ldots, i_{|\sigma|}$ such that the elements $\pi_{i_1} \pi_{i_2} \ldots \pi_{i_{|\sigma|}}$ occur in the same order as the elements of σ, and if $j \in X$ then the π_is corresponding to j and $j + 1$ in σ must differ by one.

Example

$\underline{14}2\underline{3}$ contains $(132, \{2\})$, but avoids $(123, \{1, 2\})$ because 3 and 4 are consecutive values, but 1 and 3 are not.

Flexible Schemes for Covincular Patterns

- Flexible schemes work with modifications
- Permutations (and downfixes) are *spaced* (some consecutive values are treated as non-consecutive)
- Enumerate many more classes than previously

Nonexistance of Schemes

How do we know that no (traditional) enumeration scheme exists?

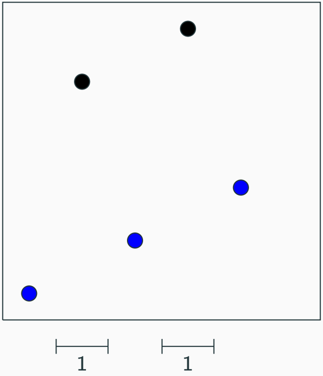
Theorem

Let B be a set of patterns whose longest increasing interval has length l. Suppose that $\lfloor k/2 \rfloor - 1 \geq (l-1)(||B||_\infty - 1) + l$, and $\pi' = 12\ldots(k-1)$ is ES-irreducible for B. Then $\pi = 12\ldots k$ is also ES-irreducible for B.

If B's longest increasing interval has length 1 and 123 is ES-irreducible, then B has no scheme

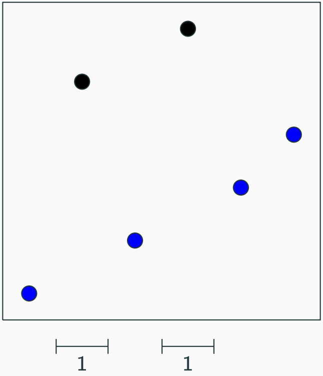
Proof Idea

$B = \{1324\}$, show 1 in 123 is ES-irreducible



Proof Idea

$B = \{1324\}$, show 1 in 1234 is ES-irreducible



Future Work

- Flexible schemes for words
- Flexible schemes for classes with few simple permutations
- Enumeration schemes + structure paradigm (e.g. next section)

1423- (equivalently 1342-) avoiding permutations

History

- Sequence $|\mathrm{Av}(1342; n)|$ has g.f. $\frac{32x}{1+20x-8x^2-(1-8x)^{3/2}}$
- Bóna (1997)
- Bloom and Elizalde (2013)
- We show recurrence using ES and structure

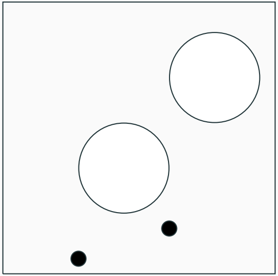
Preliminary Observations

Observation

Downfix 21 is ES-reducible (delete 1)

Observation

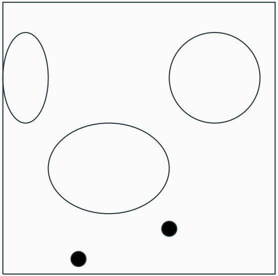
If π avoids 1423 and has a 12 downfix, then all the elements in π occurring between the 1 and 2 are smaller than all the elements occurring after the 2.



A Structure Lemma (1)

Lemma

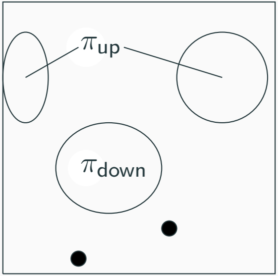
For all $\pi \in A(\{1423\}, 12, [i,j], n)$ *there exists* $k \leq i$ *such that* $\{\pi_k, \pi_{k+1}, \ldots, \pi_j\} = \{1, 2, \ldots, j-k+1\}$*; that is the* $j-k+1$ *elements between* π_k *and* π_j *are smaller than all other elements of* π.



A Structure Lemma (2)

Lemma

Let π_{down} and π_{up} be defined as per the picture, and let $\pi_{up'}$ be the reduced subpermutation of π consisting of all the elements with indices not in $[k+1, j]$ (so $\pi_{up'}$ has one more element than π_{up}). Then, π avoids 1423 if and only if π_{down} and $\pi_{up'}$ both avoid 1423.



Avoiding Double Counting

- Try multiplying number of π_{down}s by number of π_{up}s
- Counts perms once for each way of decomposing into π_{down} and π_{up}
- Solution: count perm once when the first component of π_{up} is as large as possible

Resulting Recurrences

$$|A(\{1423\}, \emptyset, [], n)| = \sum_{i=1}^{n} |A(\{1423\}, 1, [i], n, n)|$$

$$|A(\{1423\}, 1, [i], n, p)| = \begin{cases} \sum_{j=n-p}^{i-1} |A(\{1423\}, 1, [j], n-1, p)| & \text{if } p \geq 2 \text{ or } (p = 1, n \neq i) \\ 0 & \text{otherwise} \end{cases}$$
$$+ \begin{cases} \sum_{j=1}^{n-p-1} |A(\{1423\}, 1, [j], n-1, n-j)| & \text{if } p \geq 2 \text{ or } (p = 1, n \neq i) \\ 0 & \text{otherwise} \end{cases}$$
$$+ \sum_{j=i+1}^{n} |A(\{1423\}, 12, [i, j], n, p-1)|$$

$$|A(\{1423\}, 12, [i, j], n, p)| = \sum_{k=1}^{n-p-1} |A(\{1423\}, 1, [i-k+1], j-k, j-i-1)|$$
$$\cdot |A(\{1423\}, 1, [k], n-j+k, n-j+1)|$$
$$+ \sum_{k=n-p}^{i} |A(\{1423\}, 1, [i-k+1], j-k, j-i-1)|$$
$$\cdot |A(\{1423\}, 1, [k], n-j+k, p-j+k)|$$

A New Quantity Counted by OEIS Sequence A006012

Dashed (Vincular) Patterns

Definition

A permutation π contains a dashed pattern σ if there exists an increasing sequences $i_1, i_2, \ldots i_k$ such that $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_3}$ has the same relative order as σ and $i_j = i_{j+1}$ whenever there is no dash between σ_j and σ_{j+1}.

Example

2<u>51</u>3<u>46</u> contains 3-1-24

Example

251346 avoids 21-3-4

Callan's Conjecture

- $a_1 = 1$
- $a_2 = 2$
- $a_n = 4a_{n-1} - 2a_{n-2}, n \geq 3$

Conjecture (Callan, 2014)

$$a_n = |\,\mathrm{Av}(\{1\text{-}32\text{-}4, 1\text{-}42\text{-}3, 2\text{-}31\text{-}4, 2\text{-}41\text{-}3\}; n)|$$

De-dashing

- $A = \{1\text{-}32\text{-}4, 1\text{-}42\text{-}3, 2\text{-}31\text{-}4, 2\text{-}41\text{-}3\}$
- $B = \{1\text{-}3\text{-}2\text{-}4, 1\text{-}4\text{-}2\text{-}3, 2\text{-}3\text{-}1\text{-}4, 2\text{-}4\text{-}1\text{-}3\}$
- (or $B = \{1324, 1423, 2314, 2413\}$ as regular patterns)

Lemma

$\mathrm{Av}(A) = \mathrm{Av}(B)$

Four Ways to Add An Element

- Recall $B = \{1324, 1423, 2314, 2413\}$
- Every permutation in $\mathrm{Av}(B)$ has 1 and 2 consecutive or 1 or 2 at the end
- To make a perm in $\mathrm{Av}(B)$ longer, increase each element by 1 and then apply one of these
  - f_{before} inserts 1 in permutation before 2
  - f_{after} inserts 1 in permutation after 2
  - f_{end} adds 1 at end of permutation
  - f_{bump} replaces 2 with 1, adds 2 at end of permutation

Resulting Recurrence

- Each $\pi \in \text{Av}(B; n)$ gives 4 in $\text{Av}(B; n+1)$
- Each $\pi \in \text{Av}(B; n)$ gives two perms in $\text{Av}(B; n+2)$ which are double counted
  - E.g. $34512 = f_{\text{before}}(2341)$ and $f_{\text{bump}}(2341)$
- $\text{Av}(B; n+2) = 4 \cdot \text{Av}(B; n+1) - 2 \cdot \text{Av}(B; n)$.

A Generalization of the “Raboter” Operation

Lenormand's Raboter Operation

- Write binary representation of n
- Reduce length of each run by 1
- Result is binary expansion for $r(n)$

Example

$$r(12) = 2 \qquad (1100 \mapsto 10)$$

Sums of $r(n)$

Definition

$$L(k) = \sum_{n=2^k}^{2^{k+1}-1} r(n)$$

Theorem (Conj. Sloane, Proved Zeilberger, Wu)

$$L(k) = 2 \cdot 3^{k-1} - 2^{k-1}$$

More General Bases

- Write representation of n in base b
- Reduce length of each run by 1
- Result is base b representation of $r(b, n)$

Definition

$$L(b, k) = \sum_{n=b^k}^{b^{k+1}-1} r(b, n)$$

Closed form for $L(b, k)$

Theorem

$$L(b, k) = \frac{b(b-1)}{2b-1}(2b-1)^k - \frac{b-1}{2}b^k$$

Proof Strategy

- Separate numbers with length-$(k+1)$ expansions into cases:
  - Last two digits same
  - Last two digits distinct
- Derive recurrence
  $L(b,k) = (2b-1)\cdot L(b,k-1) + b^{k-1}\frac{(b-1)^2}{2}$ for $k \geq 2$.
- $\frac{b(b-1)}{2b-1}(2b-1)^k - \frac{b-1}{2}b^k$ has same recurrence and initial conditions.

Higher Powers

Definition

$$L(p, b, k) = \sum_{n=2^k}^{2^{k+1}-1} r(b, n)^p$$

Definition

$$L(l, p, b, k) = \sum_{2^k \leq n \leq 2^{k+1}-1, n \text{ ends in digit } l} r(b, n)^p$$

Recurrences

$$
\begin{aligned}
L(p, b, k) = (b^p + b - 1)L(p, b, k-1) \\
+ \sum_{l=0}^{b-1} \sum_{i=1}^{p} b^{p-i} l^i \binom{p}{i} L(l, p-i, b, k-1)
\end{aligned}
$$

$$
\begin{aligned}
L(l, p, b, k) = (b^p - 1) \cdot L(l, p, b, k-1) \\
+ L(p, b, k-1) \\
+ \sum_{i=1}^{p} l^i b^{p-i} \binom{p}{i} L(l, p-i, b, k-1)
\end{aligned}
$$

Maple Package

- `raboter.txt`
- Available at
  `https://sites.math.rutgers.edu/~yb165/raboter.txt`
- Finds closed form for $L(p, b, k)$ with b, p fixed and k variable
- Conjectures closed form for $L(p, b, k)$ with p fixed, k, b variable

Examples

Theorem

$$L(2,2,k) = \frac{2}{3}5^k - \frac{1}{6}2^k - \frac{2}{3}3^k$$

Conjecture

$$\begin{aligned}L(2,b,k) = &\Big(\frac{1}{6}b^2 - \frac{1}{6}b - \frac{1}{3}\Big)(b-1)^k + \Big(-\frac{1}{6}b^2 + \frac{1}{3}b - \frac{1}{6}\Big)b^k \\ &- \frac{b(b-1)}{2b-1}(2b-1)^k + \frac{2b^3+3b^2-3b-2}{6(b^2+b-1)}(b^2+b-1)^k.\end{aligned}$$

Conclusion

- Experimental Math is a powerful enumerative tool
- Computers can rigorously count many avoidance classes
- Could count even more if we combined structural and enumeration scheme approaches

Questions?