THE ENUMERATION AND STRUCTURE OF PERMUTATION CLASSES

BY VINCENT VATTER

A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

Doron Zeilberger

and approved by

____

____

____

____

____

New Brunswick, New Jersey

January, 2006

ABSTRACT OF THE DISSERTATION

The enumeration and structure of permutation classes

by Vincent Vatter

Dissertation Director: Doron Zeilberger

Permutation classes arise naturally in many disparate fields, ranging from the analysis of sorting machines to the study of Schubert varieties. We characterize the finitely based permutation classes with finitely labeled generating trees, and describe how to systematically enumerate these classes. We then extend Zeilberger's enumeration schemes, which provide another powerful technique for the systematic enumeration of permutation classes. Finally, we study grid classes, characterize the permutation classes that lie in a grid class, and use this to derive a new proof of the Fibonacci dichotomy for permutation classes.

Table of Contents

Chapter 1

Motivation and examples

1.1 Crucial definitions

Two sequences of natural numbers of the same length, say $u = u(1), u(2), \ldots, u(k)$ and $w = w(1), w(2), \ldots, w(k)$, are said to be *order isomorphic* if they have the same pairwise comparisons, that is, if $u(i) < w(i) \iff u(j) < w(j)$. For example, $9, 1, 6, 7, 2$ is order isomorphic to $5, 1, 3, 4, 2$. Every sequence w of natural numbers without repetition is order isomorphic to a unique permutation that we denote by $\mathrm{st}(w)$, so $\mathrm{st}(9, 1, 6, 7, 2) = 5, 1, 3, 4, 2$, which we shorten to 51342. We say that $\mathrm{st}(w)$ is the *standardization* of w. We further say that the permutation π *contains* the permutation β if π contains a subsequence that is order isomorphic to β, and in this case we write $\beta \le \pi$. For example, 391867452 contains 51342, as can be seen by considering the subsequence 91672. This fact can also be seen from Figure 1.1, where the two permutations are plotted. One permutation is said to *avoid* another if it does not contain it.

This defines a partial order on the set of all permutations, which is (largely) the subject of this thesis.

1.2 Sorting

We begin, much as the field itself, with sorting. Given a permutation, which we will think of as an ordered list of values, $\pi = \pi(1)\pi(2)\cdots\pi(n)$, we would like to pass it through a sorting machine and receive the identity permutation $12\cdots n$ in return. We confine our discussion to two sorting devices: stacks and deques, but even with regard to these simple devices many intriguing questions remain. For more comprehensive surveys, we refer to Bóna [33, 34]

1.2.1 Sorting with a single stack

A *stack* is a last-in first-out linear sorting device with two operations. At any time we can *push* the next unread entry of the input permutation onto the stack, which will put it at the top of the stack. Or we can *pop* the top-most entry off the stack, making it the next symbol of the output permutation. For example, one way to sort the permutation 2143 is shown in Figure 1.2.

Not all permutations can be sorted with a single stack. For example, 231 is not stack-sortable. In fact, consider any permutation that contains 231, or in other words, any permutation of the form $\cdots b \cdots c \cdots a \cdots$ where $a < b < c$. When we come to the element b we push it onto the stack, where it must remain at least until a has been popped from the stack. But then when we come to the element c we must push it onto the stack on top of b, and thus we cannot pop b off the stack before c, so none of these permutations are stack-sortable.

Sorting with a single stack can be made deterministic by using the *greedy algorithm*. The greedy algorithm for stack-sorting the permutation $\pi = \pi(1)\pi(2)\cdots\pi(n)$ goes as follows. First we push $\pi(1)$ onto the stack. Now suppose at some later stage that the letters $\pi(1), \ldots, \pi(i-1)$ have all been either output or pushed on the stack, so we are reading $\pi(i)$. We push $\pi(i)$ onto the stack if and only if $\pi(i)$ is less than every element on the stack[1]. Otherwise we pop elements off the stack until $\pi(i)$ is less than every remaining stack element and then push $\pi(i)$ onto the stack. This algorithm produces a unique permutation $s(\pi)$.

Observation 1.2.1. *The permutation $\pi \in S_n$ is stack-sortable if and only if $s(\pi) = 12\cdots n$.*

Note that the greedy algorithm as we have defined it doesn't require a permutation as input; the same definition works for all words with distinct entries. (And indeed, it also works for permutations of a multiset.)

[1]When performing the greedy stack-sorting algorithm, the entries in the stack will be in increasing order from top-to-bottom, so this condition can be checked merely by comparing $\pi(i)$ to the element on top of the stack.

Write $\pi \in S_n$ as $\pi_1 n \pi_2$ for words π_1 and π_2. In applying the greedy algorithm to π, we will reach a point when n is the next symbol to be read, and then we will have to empty the stack before pushing n. We then push n onto the stack, where it remains until all the other entries (the entries in π_2) are dealt with. Our last operation is to pop n off the stack. This justifies our next observation.

Observation 1.2.2. *Let* $\pi = \pi_1 n \pi_2 \in S_n$. *Then* $s(\pi) = s(\pi_1)s(\pi_2)n$.

We now have all the pieces in place to prove the first result on sortability.

Theorem 1.2.3 (Knuth [89])**.** *The permutation* $\pi \in S_n$ *is stack-sortable if and only if it avoids* 231.

Proof. We have already observed that 231-containing permutations are not stack-sortable. For the other direction we use induction on n, noting that the theorem is trivially true for the empty permutation. If $\pi \in S_n$ avoids 231 then $\pi = \pi_1 n \pi_2$ where every element in π_1 is less than any element in π_2 and both π_1 and π_2 avoid 231. Then we are done by induction and Observation 1.2.2. □

1.2.2 Sorting with a deque

A *deque* ("double-ended queue") is a linear sorting device in which elements may be pushed onto or popped from either end of list. For example, one way to sort 3241 with a deque is shown in Figure 1.3.

Theorem 1.2.3 gave a very simple test for stack-sortability; to check if π is stack-sortable we only need to check if $231 \leq \pi$. On the contrary, Pratt showed that deque-sortability is more complicated.

Theorem 1.2.4 (Pratt [120])**.** *The are infinitely many minimal (w.r.t.* $\leq$*) non-deque sortable permutations.*

Thus the minimial non-deque sortable permutations form an antichain under $\leq$, and this is one of the first published examples of an infinite antichain of permutations. Another infinite antichain from around the same time can be found in Tarjan [150], while Kruskal [93] mentions that such an antichain exists but does not give a construction.

We have chosen to omit the list of minimal non-deque sortable permutations, which can be found in Pratt [120]. We will instead construct simpler examples of infinite antichains of permutations in Section 1.8.

Even though there are infinitely many minimal non-deque-sortable permutations, one can decide the membership problem (is π deque-sortable?) in linear-time, see Knuth [89]. The enumeration problem remains unsolved to this day.

Problem 1.2.5. *Find a formula for the number of deque-sortable permutations of length n.*

1.2.3 Sorting with more than one stack in series

There are several different interpretations of what it means to sort with more than one stack in series. West [157] defines the permutation π to be t-stack-sortable if $s^t(\pi)$ – the result of applying the greedy algorithm t times to π – is the identity. We will call such permutations *West-t-stack-sortable*, and denote the number of West-t-stack-sortable permutations of length n by $w_t(n)$. The definition of West-1-stack-sortability coincides with that of stack-sortability, so $w_1(n)$ is the number of 231-avoiding permutations of length n, which we will observe soon (Proposition 1.4.1) is the nth Catalan number.

West-2-stack-sortable permutations are more complicated. Note that 35241 is West-2-stack-sortable, while 3241, which is contained in 35241, is not. In fact, West-2-stack-sortable permutations can be described as the permutations that avoid 2341 and do not contain 3241 except possibly as part of a copy of 35241 (this is due to West [157]). Nevertheless, they do have a nice formula, which was conjectured by West and first proved by Zeilberger.

Theorem 1.2.6 (Zeilberger [165]; Dulucq, Gire, and West [55]; Goulden and West [75])**.** *The number of West-2-stack-sortable permutations is*

$$\frac{2(3n)!}{(n+1)!(2n+1)!},$$

sequence A000139 in the OEIS [131].

A formula for $w_3(n)$ has yet to be found. In fact, as pointed out by Bóna [33], it is not known whether $w_3(n)$ is smaller or larger than $\binom{4n}{n}$.

Other stack-sorting algorithms are possible, and for more than one stack, the iterated greedy algorithm used in West-stack-sorting is not the optimal algorithm. By allowing any sorting algorithm we reach the definition of *general t-stack-sortability*, or simply *t-stack-sortability*.

Tarjan [150] observed that all permutations of length 6 are 2-stack-sortable while 2435761 is not 2-stack-sortable. Atkinson, Murphy, and Ruškuc [14] reported that they had found 22 minimal non-2-stack-sortable permutations of length 7, 51 of length 8, and 146 of length 9 (the number of these permutations is sequence A111576 in the OEIS [131]). They conjectured that, as with deque-sorting, there are infinitely many minimal obstructions to 2-stack-sortability.

Conjecture 1.2.7 (Atkinson, Murphy, and Ruškuc [14])**.** *There are infinitely many minimal non-2-stack-sortable permutations.*

They went on to make the following bolder conjecture, which would show a somewhat surprising contrast between sorting with 2 stacks and sorting with a deque.

Conjecture 1.2.8 (Atkinson, Murphy, and Ruškuc [14])**.** *It is NP-complete to determine if a permutation can be sorted by 2 stacks in series.*

Another difficult problem is to find the shortest non-t-stack-sortable permutation. In one direction, we have the following result of Tarjan.

Theorem 1.2.9 (Tarjan [150])**.** *For $t \geq 2$, all permutations of length $3 \cdot 2^{t-1}$ or less are t-stack-sortable.*

This gives a lower bound of 13 for the length of the shortest permutation that cannot be sorted with 3 stacks in series. Tarjan mentions that he constructed a permutation of length 41 that could not be sorted with 3 stacks in series, while Murphy [112] gives an example of length 39. Thus the following question remains open.

Question 1.2.10 (Tarjan [150])**.** *What is the shortest permutation that cannot be sorted by 3 stacks in series?*

1.3 Permutation classes in general

In concluding his discussion of sorting machines, Pratt [120] remarks "from an abstract point of view, the [containment order] on permutations is even more interesting than the networks we were characterizing." This opinion is shared by the present author, and so upon concluding our brief discussion of sorting machines, we now move on to studying the containment order in the abstract.

A *permutation class* (or *closed class of permutations*) is a lower order ideal in the containment order, meaning that if π is contained in a permutation in the class, then π itself lies in the class. Thus all of the sets of permutations we have seen so far, except West-2-stack-sortable permutations, have been permutation classes. Permutation classes can be specified in terms of the minimal permutations not lying in the class, which we call the *basis* of the class. By this minimality condition, bases are necessarily *antichains*, meaning that no element of a basis is contained in another. As we saw with Theorem 1.2.4, there are infinite antichains of permutations; we will frequently need to restrict our attention to *finitely based* classes, i.e., classes with finite bases.

Given a set of permutations B, we define $\mathrm{Av}(B)$ to be the set of permutations that avoid all of the permutations in B. Thus if $\mathcal{C}$ is a permutation class with basis B then $\mathcal{C} = \mathrm{Av}(B)$, and for this reason the elements of a permutation class are often referred to as *restricted permutations* or *pattern-avoiding permutations*. We also make the following definitions:

- $s_n(\mathcal{C})$ denotes the number of n-permutations in $\mathcal{C}$,
- $\mathrm{Av}_n(B)$ denotes the set of permutations of length n in $\mathrm{Av}(B)$,
- $s_n(B)$ denotes the number of n-permutations in $\mathrm{Av}(B)$,
- the generating function for $\mathcal{C}$ is $\displaystyle\sum_{n\geq 0} s_n(\mathcal{C})x^n$.

About this last definition: note that we start this sum at 0, thereby including the empty permutation, which we denote by $\emptyset$. This choice seems to be a nuisance (e.g., Proposition 1.5.2) about as often as it is a convenience (e.g., Proposition 1.4.1).

In this language the basis for the stack-sortable permutations is $\{231\}$, while the deque-sortable permutations are infinitely based, and the basis for the 2-stack-sortable permutations is unknown, but is conjectured to be infinite.

Before moving on to examples we must say a word about symmetries. Note that $\sigma \leq \pi$ if and only if $\sigma^{-1} \leq \pi^{-1}$, where here $^{-1}$ denotes the group-theoretic inverse. Similarly, $\sigma \leq \pi$ if and only if the reverse of σ is contained in the reverse of π, where the reverse of $\pi = \pi(1)\pi(2)\cdots\pi(n)$ is $\pi(n)\cdots\pi(2)\pi(1)$. These two operations generate the dihedral group with 8 elements.

1.4 Examples of permutation classes I (small bases)

1.4.1 Avoiding a permutation of length 3

It is a classic result that $s_n(\beta)$ is the nth Catalan number, C_n, for all permutations $\beta \in S_3$. By symmetry, $s_n(132) = s_n(213) = s_n(231) = s_n(312)$ and $s_n(123) = s_n(321)$, so this really only incorporates one fact: $s_n(231) = s_n(123)$. A bijective proof of this result was given by Simion and Schmidt [130] (see also Bóna [34, Section 4.2]). Zeilberger [164] gives a proof using a technique quite like the enumeration schemes we discuss in Chapter 5 that generalizes to permutations of a multiset. Here we content ourselves with verifying that $s_n(231) = C_n$.

Proposition 1.4.1. *The number of* 231*-avoiding permutations in* S_n *is the* n*th Catalan number.*

Proof. As observed in the proof of Theorem 1.2.3, every permutation $\pi \in \mathrm{Av}(231)$ can be written as $\pi_1 n \pi_2$ where π_1 and π_2 avoid 231 and every element of π_2 is greater than every element of π_1 (see Figure 1.4), and conversely, every permutation of this form avoids 231. This shows that the generating function f for $\mathrm{Av}(231)$ satisfies

$$f = 1 + xf^2,$$

and therefore establishes that $s_n(231)$ is the nth Catalan number. □

Although $\mathrm{Av}(123)$ and $\mathrm{Av}(231)$ are equinumerous, $\mathrm{Av}(231)$ has a much simpler structure than Av(123). For example, Av(123) contains an infinite antichain (the reverse of the antichain U from Section 1.8) while Av(231) does not (this is proved in Atkinson, Murphy, and Ruškuc [13]). Moreover, the enumeration problem for subclasses of Av(231) has a very nice answer:

Theorem 1.4.2 (Albert and Atkinson [3])**.** *Every proper subclass of* Av(231) *has a rational generating function.*

Theorem 1.4.2 is made more impressive by the fact that its proof is constructive. The analogue for 123-avoiding permutations remains open.

Problem 1.4.3. *Find a method to compute the generating function for all finitely based subclasses of* Av(123).

1.4.2 Permutations without long increasing subsequences

Permutations avoiding a monotone pattern, which without loss of generality we may take to be $12 \cdots k$, have been extensively studied. One of the first general results, due to Regev [124], states that

$$s_n(12 \cdots k) \sim \lambda_k \frac{(k-1)^2}{n^{(k^2-2k)/2}}, \tag{1.1}$$

where λ_k is a specific multiple integral. A decade later Gessel [74] found explicit generating functions for the classes $\text{Av}(12 \cdots k)$. For $k = 3$, this generating function is algebraic (it is the generating function for the Catalan numbers), but for $k \geq 4$ it is not (although it is holonomic, which is a term defined in Section 2.5). For $k = 4$ he gives the formula (see also Stanley [145, Exercise 7.12.e])

$$s_n(1234) = \frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}. \tag{1.2}$$

(Sequence A005802 in the OEIS [131].)

Stanley [146] expressed the importance of Gessel's result as follows:

> Gessel's theorem reduces the theorems of Baik, Deift, and Johansson[2] to

[2] This refers to their theorem from [19] stating that the distribution of the longest increasing subsequence statistic on S_n converges in distribution, after rescaling, to the Tracy-Widom distribution.

"just" analysis, viz., the Riemann-Hilbert problem in the theory of integrable systems, followed by the method of steepest descent to analyze the asymptotic behaviors of integrable systems.

1.4.3 Avoiding 1324

Marinov and Radoičić [106] used a generating tree approach (see Section 4.1 for a treatment of generating trees) to find the first 20 terms of $s_n(1324)$; these are given by sequence A061552 in the OEIS [131]. Recent work by Albert, Elder, Rechnitzer, Westcott and Zabrocki [1] gives another six terms for the sequence $s_n(1324)$, but no one has yet succeeded in finding a formula for $s_n(1324)$. Moreover, some feel that no nice formula for this class exists, see Section 2.5.

Arratia [10] wagered \$100 on the conjecture that $s_n(\beta) \leq (k-1)^{2n}$ for all $\beta \in S_k$, and Bóna [35] subsequently made an even stronger (but prizeless) conjecture. Both were disproved by Albert et al., who proved that $s_n(1324) > (9.35)^n$ for sufficiently large n. The lowest known upper bound is $s_n(1324) < 288^n$, see Bóna [34, Theorem 4.30].

There is one more interesting result about this class. Albert, Aldred, Atkinson, van Ditmarsch, Handley, and Holton [2] introduce a sorting machine called a *loosely locked jump queue* that can generate precisely Av(1324).

1.4.4 The Erdős-Szekeres Theorem

In 1935, Erdős and Szekeres observed that the monotone permutations are unavoidable:

The Erdős-Szekeres Theorem [61]. *For all* $\pi \in \mathrm{Av}(12\cdots j, k\cdots 21)$,

$$|\pi| < (j-1)(k-1)+1.$$

While proofs of this theorem are not in short supply, we give another. It could be argued that the proof presented here is essentially a reformulation of the proofs given by Blackwell [28] and Hammersley [79].

Given a permutation π, we say that the element $\pi(r)$ is a *right-to-left maximum* of π if $\pi(s) < \pi(r)$ for all $s > r$.

Proof of the Erdős-Szekeres Theorem: We use induction on $j + k$. The theorem is trivial if $j = 1$ or $k = 1$ because only the empty permutation avoids 1. Because of this observation and symmetry, we may assume that $j \geq k > 1$. Take $\pi \in \text{Av}(12 \ldots j, k \ldots 21)$, and let σ denote the permutation gotten from π by removing every right-to-left maximum in π and standardizing.

Because the right-to-left maximums of π form a decreasing subsequence there can be at most $k-1$ of them, so $|\sigma| \geq |\pi| - (k-1)$. Furthermore, every maximal increasing subsequence in π must end at a right-to-left maximum (otherwise it could be extended), so $\sigma \in \text{Av}(12 \ldots (j-1), k \ldots 21)$. By induction then

$$|\pi| - (k-1) \leq |\sigma| < (j-2)(k-1) + 1,$$

completing the proof. □

As a corollary we have our first (and admittedly very modest) result connecting the structure of a class to its basis.

Corollary 1.4.4. *The permutation class $\mathcal{C}$ is finite if and only if its basis contains both an increasing permutation and a decreasing permutation.*

Proof. If the basis of $\mathcal{C}$ does not contain an increasing (resp. decreasing) permutation, then $\mathcal{C}$ contains $12 \cdots n$ (resp. $n \cdots 21$) for all n, so $\mathcal{C}$ is not finite. On the other hand, if the basis of $\mathcal{C}$ contains $12 \cdots j$ and $k \cdots 21$ then $\mathcal{C} \subseteq \text{Av}(12 \cdots j, k \cdots 21)$, which is finite by the Erdős-Szekeres Theorem. □

1.5 Examples of permutation classes II ($\oplus$ and $\ominus$)

Here we study operations to build new classes from old ones.

1.5.1 Direct sums and skew sums

We begin with two binary operations on permutations. Given two permutations $\pi \in S_m$ and $\sigma \in S_n$, we define their *direct sum*, written $\pi \oplus \sigma$ by

$$(\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{if } i \in [m], \\ \sigma(i-m) + m & \text{if } i \in [m+n] \setminus [m]. \end{cases}$$

Similarly, we define their *skew sum*, $\pi \ominus \sigma$, by

$$(\pi \ominus \sigma)(i) = \begin{cases} \pi(i) + n & \text{if } i \in [m], \\ \sigma(i - m) & \text{if } i \in [m+n] \setminus [m]. \end{cases}$$

Examples are shown in Figure 1.5.

A permutation is said to be $\oplus$-*indecomposable* or *sum indecomposable* if it cannot be written as the direct sum of two shorter (but nonempty) permutations, and $\ominus$-indecomposable or *skew sum indecomposable* if it cannot be written as the skew sum of two shorter nonempty permutations. Otherwise the permutation is said to be $\oplus$-decomposable or $\ominus$-decomposable, respectively. A class $\mathcal{C}$ is said to be $\oplus$-*complete* or *sum complete* if $\pi \oplus \sigma \in \mathcal{C}$ whenever $\pi, \sigma \in \mathcal{C}$. Analogously, a class is said to be $\ominus$-*complete* or *skew sum complete* if $\pi \ominus \sigma \in \mathcal{C}$ for all $\pi, \sigma \in \mathcal{C}$.

For example, the 321-avoiding permutations are $\oplus$-complete, while $\text{Av}(123, 6543217)$ is $\ominus$-complete. It is easy to decide from its basis if a finitely based classes is $\oplus$-complete or, by symmetry, $\ominus$-complete.

Proposition 1.5.1 (Atkinson and Stitt [16]). *A permutation class is $\oplus$-complete if and only if each of its basis elements is $\oplus$-indecomposable.*

Proof. First let $\mathcal{C}$ be a $\oplus$-complete class and suppose to the contrary that the basis of $\mathcal{C}$ contains a $\oplus$-decomposable permutation $\beta = \beta_1 \oplus \beta_2$ where $|\beta_1|, |\beta_2| \geq 1$. Because β is a basis element, $\beta_1, \beta_2 \in \mathcal{C}$, but $\beta \notin \mathcal{C}$, a contradiction.

To establish the other direction, suppose that every basis element of the class $\mathcal{C}$ is $\oplus$-indecomposable, but that that $\pi \oplus \sigma \notin \mathcal{C}$ for some $\pi, \sigma \in \mathcal{C}$. Then $\pi \oplus \sigma$ contains a basis element of $\mathcal{C}$, say β. This pattern cannot be fully contained in either π or σ because these permutations lie in $\mathcal{C}$, but any embedding of β which involves elements of both π and σ gives a nontrival $\oplus$-decomposition of β, a contradiction. □

From Proposition 1.5.1 and the observation that a nontrivial permutation cannot be both $\oplus$-decomposable and $\ominus$-decomposable, it follows that every singleton-based class $\text{Av}(\beta)$ is either $\oplus$-complete or $\ominus$-complete.

1.5.2 $\oplus$-completions

Given a permutation class $\mathcal{C}$, we define the $\oplus$-completion of $\mathcal{C}$, denoted by $\bigoplus \mathcal{C}$, to be the smallest $\oplus$-complete permutation class containing $\mathcal{C}$. Equivalently,

$$\bigoplus \mathcal{C} = \{\pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k : \pi_1, \pi_2, \ldots, \pi_k \in \mathcal{C}\}.$$

Analogously, we also have $\ominus$-completions, but by symmetry we never need to talk about them.

Given enough information about the class $\mathcal{C}$, it is easy to solve the enumeration problem for $\bigoplus \mathcal{C}$.

Proposition 1.5.2. *Let f denote the generating function for the set of $\oplus$-indecomposable permutations in the nonempty permutation class $\mathcal{C}$. Then the generating function for $\bigoplus \mathcal{C}$ is $1/(2-f)$.*

Proof. There is a canonical bijection between elements of $\bigoplus \mathcal{C}$ and sequences of nonempty $\oplus$-indecomposable permutations in $\mathcal{C}$. Therefore the generating function for $\bigoplus \mathcal{C}$ is

$$1 + (f-1) + (f-1)^2 + \cdots = \frac{1}{1-(f-1)},$$

as desired. □

1.5.3 Layered permutations

The *layered permutations* are the sum completion of the set of decreasing permutations, $\bigoplus\{1, 21, 321, \ldots\}$. An example is shown in Figure 1.6. Since all of the decreasing permutations are $\oplus$-indecomposable, Proposition 1.5.2 shows that the generating function for the layered permutations is $(1-x)/(1-2x)$.

Clearly 231 and 312 are not layered permutations, and it is not hard to verify that these permutations form the basis for the layered permutations. We give two short proofs of this fact below. One of these proofs uses an elementary argument. The other employs the Maple package FinLabel described in Chapter 4; until then we will treat FinLabel as a black box that can enumerate some permutation classes completely mechanically.

Proposition 1.5.3. *The layered permutations are* $\mathrm{Av}(231, 312)$.

Proof #1. We have already observed that the layered permutations are contained in $\mathrm{Av}(231, 312)$, so we need to prove that every permutation $\pi \in \mathrm{Av}_n(231, 312)$ is layered. We prove this by induction on n; since this claim is trivially true for $n \leq 2$, we will assume that $n \geq 3$. Suppose that $\pi(k) = 1$. Because π avoids 231, the entries $\pi(1)$, $\pi(2)$, $\ldots$, $\pi(k)$ must be in decreasing order, and since π avoids 312, every number between 1 and $\pi(1)$ must occur in this initial segment. This implies that $\pi = k \ldots 1 \oplus \sigma$ for some $\sigma \in \mathrm{Av}_{n-k}(231, 312)$, and thus we are done by induction. □

Proof #2. We observed above that the generating function for the layered permutations is $(1 - x)/(1 - 2x)$. We also observed that the layered permutations are contained in $\mathrm{Av}(231, 312)$. FinLabel computes that the generating function for $\mathrm{Av}(231, 312)$ is also $(1 - x)/(1 - 2x)$, so these two classes must be equal. □

More can be said about the enumeration of layered permutations. Indeed, every subclass of this class has a rational generating function by Theorem 1.4.2, and their Möbius function is computed in Sagan and Vatter [126] (see also Björner and Sagan [27]).

1.5.4 Strong completions & separable permutations

The class of separable permutations, first introduced by Bose, Buss, and Lubiw [37], is essentially the permutation analogue of series-parallel posets (see Stanley [139] or [144, Section 3.2]) and complement reducible graphs (see Corneil, Lerchs, and Burlingham [49]). To define them we need to introduce the notion of strong completion.

Given a class $\mathcal{C}$, we denote by $\mathrm{sc}(\mathcal{C})$ the *strong completion* of $\mathcal{C}$, which is the smallest class containing $\mathcal{C}$ such that both $\pi \oplus \sigma$ and $\pi \ominus \sigma$ lie in $\mathrm{sc}(\mathcal{C})$ for every $\pi, \sigma \in \mathrm{sc}(\mathcal{C})$. By combining both parts of Proposition 1.5.1 we get a test to decide if finitely based classes are strongly complete.

Proposition 1.5.4. *A class is strongly complete if and only if each of its basis elements is both* $\oplus$*-indecomposable and* $\ominus$*-indecomposable.*

The *separable permutations* are the strong completion of $\{1\}$. As was shown by Bose, Buss, and Lubiw [37], this class can also be described as $\text{Av}(2413, 3142)$. The enumeration of this class (which can now be done routinely with substitution decompositions, see Subsection 3.2.2) was first undertaken by West [158]. He used generating trees (described in Chapter 4) to show that the separable permutations are counted by the large Schröder numbers. Later, Ehrenfeucht, Harju, ten Pas, and Rozenberg [57] (who also gave another proof that the basis of this class is $\{2413, 3142\}$) presented a bijection between separable permutations and parenthesis words, the objects Schröder was originally interested in counting.

1.5.5 Permutations that embed into the increasing oscillating sequence

The *increasing oscillating sequence* is the infinite sequence

$$4, 1, 6, 3, 8, 5, \ldots, 2k+2, 2k-1, \ldots.$$

A plot is shown in Figure 1.7.

In this subsection we consider the set of permutations that embed into the increasing oscillating sequence, that is, the permutations that can obtained as standardizations of subsequences of this sequence. For example, $\text{st}(4, 6, 5, 10) = 1324$, so 1324 lies in this set. Clearly this set is a permutation class. It would therefore be nice to find its basis and generating function. We begin with the generating function.

Proposition 1.5.5. *The generating function for the class of all permutations that embed into the increasing oscillating sequence is*

$$\frac{1-x}{1-2x-x^3}.$$

Proof. For $n \geq 3$ there are precisely two $\oplus$-indecomposable permutations of length n that embed into the increasing oscillating sequence. For even n these are the standardization of the first n entries of the sequence and the standardization of the subsequence containing the first element, the third through nth elements, and the $n+2$nd element. For odd n these are the standardization of the subsequence containing the first element, and the third through $n+1$st elements, and the standardization of the subsequence containing the first through $n-1$st elements together with the $n+1$st element. Thus the generating function for the $\oplus$-indecomposable permutations in this class is

$$f = 1 + x + x^2 + 2x^3 + 2x^5 + \cdots + 2x^n + \cdots = \frac{1+x^3}{1-x}.$$

An application of Proposition 1.5.2 now completes the proof. □

The basis of this class is mentioned without proof in Murphy's thesis [112] while Albert, Atkinson, and Ruškuc [unpublished] found a structural proof. We present a proof which, like the second proof given for Proposition 1.5.3, uses the Maple package FinLabel. Unlike the second proof of Proposition 1.5.3, here this approach saves a significant amount of work.

Proposition 1.5.6. *The class of permutations that embeds into the increasing oscillating sequence is* $\mathrm{Av}(321, 2341, 3412, 4123)$.

Proof. Let $\mathcal{C}$ denote the set of all permutations that embed into the increasing oscillating sequence. It is not difficult to check that $\mathcal{C} \subseteq \mathrm{Av}(321, 2341, 3412, 4123)$, and we showed in Proposition 1.5.5 that the generating function for $\mathcal{C}$ is $(1-x)/(1-2x-x^3)$. FinLabel computes that this is also the generating function for $\mathrm{Av}(321, 2341, 3412, 4123)$, so the two classes are identical. □

1.6 Closed classes in other contexts

Thus far we have only discussed permutation classes, but there are compelling reasons to study closed classes[3] of all sorts of combinatorial objects, even if our interest lies mostly with permutations. For one, the tools and results from other contexts are often useful for permutation classes. For example, in Subsection 1.7.1 we derive the basis for an interesting class of permutations from a well-known theorem about graphs, while in Section 1.8 we show how the existence of permutation antichains follows immediately from easy-to-spot graph antichains, and of course, the Stanley-Wilf Conjecture (see Section 2.1) was proved by analyzing 0/1-matrices, not permutations. While the options

[3] In other contexts, such classes are often called *hereditary*.

are virtually limitless, we will discuss only a handful of other objects: vectors, words, compositions, partitions, posets, graphs, and 0/1-matrices.

We will mostly carry over our notation from permutation classes without change. For example, a *closed class* or simply *class* will be a set of objects closed downward under whatever order we are studying, and we will say the basis of the class $\mathcal{C}$ is the set of minimal objects not in $\mathcal{C}$. The class with basis B will be denoted by $\mathrm{Av}(B)$. All of the objects we will study will have a natural analogue of "length," and we will refer to the "length"-generating function for the class $\mathcal{C}$ simply as the generating function for $\mathcal{C}$.

1.6.1 Vectors

The simplest interesting context in which to study closed classes is perhaps finite vectors of nonnegative integers. Let $\mathbf{x} = (x_1, x_2, \dots, x_m), \mathbf{y} = (y_1, y_2, \dots, y_m) \in \mathbb{N}^m$ for some m. We write $\mathbf{x} \le \mathbf{y}$ if $x_i \le y_i$ for all $i \in [m]$. This order is often called the *product order*. The weight of the vector $\mathbf{x}$, denoted $\|\mathbf{x}\|$, is the sum of the entries of $\mathbf{x}$.

Let $\mathcal{C}$ denote a closed subset of $\mathbb{N}^m$. Stanley posed the following theorem as a Monthly problem in 1976.

Theorem 1.6.1 (Stanley [140]; see also Stanley [144, Problem 4.6])**.** *Let $\mathcal{C}$ denote a closed subset of $\mathbb{N}^m$. For sufficiently large n, the number of vectors in $\mathcal{C}$ of weight n is given by a polynomial.*

While this theorem can be proved easily by viewing the function in question as a Hilbert function, we give a proof that brings out some themes which we will return to later. We begin with very special closed classes. Foreshadowing a bit, let us say that the *age* of the vector $\mathbf{y} \in (\mathbb{N} \cup \infty)^m$ is the set of all vectors in $\mathbb{N}^m$ less than y:

$$\mathrm{Age}(\mathbf{y}) = \{\mathbf{x} \in \mathbb{N}^m : \mathbf{x} \le \mathbf{y}\}.$$

It is easy to prove that ages are enumerated by polynomials.

Proposition 1.6.2. *For every vector $\mathbf{y} \in (\mathbb{N} \cup \infty)^m$ and all sufficiently large n, the number of vectors in $\mathrm{Age}(\mathbf{y})$ of weight n is given by a polynomial in n.*

Proof. The weight-generating function for $\text{Age}(\mathbf{y})$ is

$$\prod_{i\in[m]} \sum_{j=0}^{y_i} q^j,$$

which is the product of a power of $1/(1-q)$ and a polynomial in q. The proposition now follows from the binomial theorem. $\square$

By Proposition 1.6.2 and inclusion-exclusion, Theorem 1.6.1 will be proved if we can show that all closed classes in $\mathbb{N}^m$ are unions of a finite number of ages. To prove this, we first need to observe that all antichains in $(\mathbb{N}\cup\infty)^m$ are finite. Recall that a poset is said to be *partially well-ordered (pwo)* if it contains neither an infinite decreasing subsequence nor an infinite antichain. The fact that $(\mathbb{N}\cup\infty)^m$ is pwo follows from the proposition below. Nash-Williams [114] may be one of the first places where this proposition appeared.

Proposition 1.6.3. *If the posets A and B are both pwo then so is their product, $A\times B$ (where $(a,b)\le(a',b')$ if and only if $a\le a'$ and $b\le b'$).*

Proof. Clearly $A\times B$ cannot contain an infinite decreasing sequence, so suppose to the contrary that $A\times B$ contains an infinite antichain, say $(a_1,b_1),(a_2,b_2),\ldots$. Because A is pwo, every infinite sequence of elements of A contains an infinite increasing subsequence (one way to see this is to invoke Ramsey's Theorem). Hence there are indices $1\le i_1 < i_2 < \cdots$ so that $a_{i_1}\le a_{i_2}\le\cdots$. But then the sequence $b_{i_1}, b_{i_2},\ldots$ cannot contain an ascent, so it must either contain an infinite decreasing sequence or an infinite antichain, both contradictions. $\square$

We can now prove our desired result, completing the proof of Theorem 1.6.1.

Proposition 1.6.4. *Every closed class $\mathcal{C}\subseteq\mathbb{N}^m$ can written as a finite union of ages.*

Proof. The class $\mathcal{C}$ can certainly be written as a union of ages, for example, $\mathcal{C} = \bigcup_{\mathbf{y}\in\mathcal{C}}\text{Age}(\mathbf{y})$. Now note that the union of a chain of ages is again an age; if $\text{Age}(\mathbf{y}^{(1)})\subseteq \text{Age}(\mathbf{y}^{(2)})\subseteq\text{Age}(\mathbf{y}^{(3)})\subseteq\cdots$ then $\bigcup_{i\in\mathbb{N}}\text{Age}(\mathbf{y}^{(i)}) = \text{Age}(\mathbf{y})$ where $\mathbf{y}=\lim_{i\in\mathbb{N}}\mathbf{y}^{(i)}$. Thus by Zorn's lemma we may write $\mathcal{C}$ as a union of maximal ages, say $\mathcal{C} = \text{Age}(\mathbf{y}^{(1)})\,\cup$

Age($\mathbf{y}^{(2)}$) $\cup \cdots$. Because these ages are maximal, $\{\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \dots\}$ must form an antichain, and since $(\mathbb{N} \cup \infty)^m$ is pwo, this antichain must be finite. □

1.6.2 Words

If A is any set then the corresponding *Kleene closure* or *free monoid*, A^*, is the set of words with letters from A, i.e.,

$$A^* = \{w = w(1)w(2)\dots w(n) : n \geq 0 \text{ and } w(i) \in A \text{ for all } i\}.$$

There are two natural orders on A^*. One can consider u to be smaller than w if w can be written as $v_1 u v_2$ for some words $v_1, v_2 \in A^*$. This is frequently called the *factor order*. We will instead consider a different order, called the *subword order* (or *division order* or *subsequence order*): we write $u \leq w$ and say that u is a subword of w if there are indices $i_1 < i_2 < \cdots < i_k$ so that $w(i_1)w(i_2)\cdots w(i_k) = u$.

Closed classes of words (under the subword ordering) always have rational generating functions. Our proof of this result follows much the same structure as the proof of the polynomial result for vectors we gave in Subsection 1.6.1. We begin by showing that all antichains in this poset are finite. This result is a special case of a theorem of Higman [80], as was Proposition 1.6.3[4].

Theorem 1.6.5 (Higman [80])**.** *The poset A^* is pwo for all finite sets A.*

Proof. We follow the proof given by Sakarovitch and Simon in Lothaire [96] which is essentially the proof given by Nash-Williams [114]. It is easy to see that A^* does not have an infinite strictly decreasing subsequence, so suppose to the contrary that there is an infinite antichain in A^*. Then there is certainly an infinite sequence of words $w_1, w_2, \dots$ such that $w_i \not\leq w_j$ whenever $i < j$ (for example, one could take an infinite antichain). We will call such a sequence a *bad sequence*. Now we construct a *minimal bad sequence* by choosing w_1 as short as possible so that it lies in a bad sequence, then w_2 as short as possible so that there is a bad sequence beginning with w_1 and w_2, then

[4]To see this, note that if A and B are both pwo, then so is their disjoint union $A \cup B$ (where $x \leq y$ if and only if either $x, y \in A$ and $x \leq_A y$ or $x, y \in B$ and $x \leq_B y$). Theorem 1.6.5 then shows that $(A \cup B)^*$ is pwo, so the subposet $A \times B$ must also be pwo.

w_3 as short as possible so that there is a bad sequence beginning with w_1, w_2, and w_3, and so on.

Because A is finite, infinitely many w_i's begin with the same letter. Suppose that a is such a letter and let $i_1 < i_2 < \cdots$ denote the indices of the words that begin with a, so $w_{i_j} = au_{i_j}$ for some words $u_{i_j} \in A^*$. However, this implies that $w_1, w_2, \ldots, w_{i_1-1}, u_{i_1}, u_{i_2}, \ldots$ is a bad sequence, contradicting the minimality of $w_1, w_2, \ldots$ and finishing the proof. □

Theorem 1.6.6. *Let A be a finite set and $\mathcal{C}$ a closed class in A^*. Then $\mathcal{C}$ has a rational generating function.*

Proof. By Theorem 1.6.5, $\mathcal{C} = \mathrm{Av}(B)$ for a finite set of words B. For every word $u = u(1)u(2)\cdots u(k)$, the set of words containing u is described by the regular expression

$$(A \setminus \{u(1)\})^*\ u(1)\ (A \setminus \{u(2)\})^*\ u(2)\ (A \setminus \{u(3)\})^*\ \cdots\ (A \setminus \{u(k)\})^*\ u(k)\ A^*,$$

so the set of words containing any word from B is also regular, as is its complement, that is, $\mathrm{Av}(B)$. □

1.6.3 Compositions

A *composition* of the integer n is an ordered sequence of positive integers that sum to n. For example, $(9, 1, 6)$ is a composition of 16. If u is a composition of n, we will also write $\|u\| = n$. We say that the composition $u = u(1)u(2)\cdots u(k)$ is contained in the composition $w = w(1)w(2)\cdots w(n)$ if there are indices $1 \le i_1 < i_2 < \cdots < i_k \le n$ so that $u(j) \le w(i_j)$ for all $j \in [k]$. Bergeron, Bousquet-Mélou, and Dulucq [25] were some of the first to study this order.

Proposition 1.6.7. *The poset of compositions under the order defined above is isomorphic to the poset of layered permutations.*

Proof. The map φ defined by

$$\varphi(u(1)u(2)\cdots u(k)) = (u(1)\cdots 21) \oplus (u(2)\cdots 21) \oplus \cdots \oplus (u(k)\cdots 21).$$

is an order-preserving bijection between compositions and layered permutations. □

Thus by Theorem 1.4.2, every closed class of compositions has a rational generating function. Björner and Sagan found a pleasant formula for the generating functions of singleton-based classes.

Theorem 1.6.8 (Björner and Sagan [27])**.** *Let u denote a composition with ℓ_k parts equal to k for all $k \geq 1$ (so u is a composition of $\ell_1 + 2\ell_2 + 3\ell_3 + \cdots$). The generating function for* $\mathrm{Av}(u)$ *is given by*

$$\sum_{w \in \mathrm{Av}(u)} x^{\|w\|} = \frac{1-x}{1-2x}\left(1 - \prod_{k \geq 1}\left(\frac{x^k}{1-2x+x^k}\right)^{\ell_k}\right).$$

This is just one of a number of partial orders on compositions that have been studied. There is a natural analogue of the factor order has been studied by Snellman [132, 133]. The order defined above is therefore a refinement[5] of this analogue of the factor order. Additionally, Savage and Wilf [127] study compositions that avoid permutations.

1.6.4 Partitions

A *partition* of the positive integer n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of positive integers, written in nonincreasing order, that sum to n. To indicate that λ is a partition of n we write $\lambda \vdash n$. The *Ferrers diagram* of the partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is an array with k rows of empty boxes, where the ith row contains λ_i boxes. The *conjugate* of λ^* of λ is the partition whose Ferrers diagram is the transpose of λ's Ferrers diagram. Figure 1.8 shows two examples.

There is a natural partial order on partitions: we write $\kappa \leq \lambda$ if the Ferrers diagram of κ is contained in the Ferrers diagram of λ. The set of all partitions (of all sizes) equipped with this order forms a lattice known as *Young's lattice*.

1.6.5 Posets

We have been studying partially ordered sets (posets) of combinatorial objects, but since posets are themselves a combinatorial object, it is natural to also consider the poset of posets. First we must define the ordering we will take on poset: the poset

[5] The order $\preceq$ is said to be a refinement of $\leq$ if $x \leq y$ implies $x \preceq y$ for all x and y.

$Q = (X, \leq_Q)$ is contained in the poset $P = (Y, \leq_P)$ if there is an injection $\varphi : X \to Y$ so that $x_1 \leq_Q x_2$ if and only if $\varphi(x_1) \leq_P \varphi(x_2)$.

To every permutation $\pi \in S_n$ we can associate a poset, which we will denote by P_π. The elements of this poset are $[n]$ and $i \leq_{P_\pi} j$ if and only if $i \leq j$ and $\pi(i) \leq \pi(j)$ (as integers). Figure 1.9 shows two examples. This poset has proved to be a useful object in the study of *packing densities* of permutations. We will not delve into this topic, but instead refer the reader to Albert, Atkinson, Handley, Holton, and Stromquist [4].

1.6.6 Graphs

There are a wide range of orderings on graphs to study, but we focus exclusively on the induced subgraph order because it is the most analogous to the containment order on permutations.

We begin by setting some notation. Let G be a graph with vertices $V(G)$ and edges $E(G)$. For any subset $U \subseteq V(G)$ of vertices, the *subgraph induced by* U, written $G[U]$, has U as its vertices and $E(G) \cap U^2$ as its edges. The graph H is said to be an *induced subgraph* of G, which we will write as $H \leq G$, if H is isomorphic to $G[U]$ for some set $U \subseteq V(G)$. Strictly speaking, this order on graphs is a quasi-order rather than a partial order (it lacks antisymmetry), but whenever this distinction matters we will simply talk about isomorphism classes of graphs, which are partially ordered.

For any permutation $\pi \in S_n$ we define the *permutation graph* of π, G_π, to be the graph with vertex set $[n]$ where $i \sim j$ if and only if either $i < j$ and $\pi(i) > \pi(j)$ or $j < i$ and $\pi(j) > \pi(i)$, so a more descriptive name for G_π would be the inversion graph of π. These graphs are particularly easy to draw from the plot of a permutation, see Figure 1.10. (Also note that the permutation graph of π is the comparability graph of the poset of the reverse of π.)

Of the observations one may make about the correspondence between permutations and permutation graphs, four are particularly relevant:

- Permutation graphs for distinct permutations may be isomorphic. In fact, $G_\pi \cong G_{\pi^{-1}}$ for all permutations π.

- If $\sigma \leq \pi$ then G_σ is an induced subgraph of G_π, but by our first observation, the converse does not hold in general.
- Increasing subsequences in π correspond to independent sets in G_π.
- Decreasing subsequences in π correspond to cliques in G_π.

Another class of graphs are the *split graphs*, which we use to derive a result about permutation classes in Subsection 1.7.1. A graph G is said to be split if its vertices can be partitioned as $V(G) = V_1 \uplus V_2$ so that $G[V_1]$ is complete and $G[V_2]$ is edgeless. Clearly the set of split graphs is a closed class with respect to the induced subgraph order; its basis is given by the theorem below.

Theorem 1.6.9 (Földes and Hammer [66])**.** *The split graphs are* $\mathrm{Av}(K_2 \uplus K_2, C_4, C_5)$.

1.6.7 0/1-matrices

First we need to establish the notation we will use for matrices. Let M be an $m \times n$ matrix. We will denote the entry of M in row i and column j by $M_{i,j}$. A *submatrix* of M is a matrix obtained by selecting specific rows and columns from M. If $I \subseteq [m]$ is a selection of rows and $J \subseteq [n]$ is a selection of columns, we denote by $M_{I \times J}$ the submatrix $(M_{i,j})_{i \in I, j \in J}$. We define the *support* of M, written $\mathrm{supp}(M)$, to be the set of pairs (i,j) for which $M_{i,j} \neq 0$.

Given two 0/1-matrices M and N, we say that N *contains an M-pattern*, and write $M \leq N$, if N contains a submatrix whose support contains the support of M. For example,

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \leq \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \mathbf{1} & 0 & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 0 & 1 & 0 \\ \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

as can be seen by considering the entries in bold.

To continue the parallel with the other contexts, if B is a set of 0/1-matrices, let us define $\mathrm{Av}(B)$ to be the set of B-avoiding 0/1-matrices (of all shapes and sizes), and

$\mathrm{Av}_{m\times n}(B)$ to be the set of B-avoiding $m \times n$ 0/1-matrices. We further define $s_{m\times n}(B)$ to be the cardinality of $\mathrm{Av}_{m\times n}(B)$.

However, in the context of 0/1-matrices, extremal functions are much more commonly studied. For a set B of 0/1-matrices we let $\mathrm{ex}(n, B)$ denote the maximum number of 1's in a B-avoiding $n \times n$ 0/1-matrix. The first systematic study of $\mathrm{ex}(n, B)$ was initiated by Füredi and Hajnal [73].

To every permutation $\pi \in S_n$ there is a corresponding $n\times n$ 0/1 *permutation matrix*, M^π. This matrix has $\mathrm{supp}(M^\pi) = \{(i, \pi(i)) : i \in [n]\}$. Clearly $\sigma \leq \pi$ if and only if $M^\sigma \leq M^\pi$.

1.7 Examples of permutation classes III

Here we present two more well-studied permutation classes. Both are special cases of the grid classes we introduce in Chapter 7.

1.7.1 Skew-merged permutations

A permutation is said to be skew-merged if it is the union of an increasing subsequence and a decreasing subsequence. For example, the permutation shown in Figure 1.11 is a skew-merged permutation. Stankova [138] was the first to find the basis of this class. We present the proof due to Kézdy, Snevily, and Wang [86].

Theorem 1.7.1 (Stankova [138]; Kézdy, Snevily, and Wang [86]; and Atkinson [11])**.** *The skew-merged permutations are* $\mathrm{Av}(2143, 3412)$.

Proof. Clearly 2143 and 3412 are not skew-merged permutations, so we only need to prove that every permutation in $\mathrm{Av}(2143, 3412)$ is skew-merged. Take $\pi \in \mathrm{Av}(2143, 3412)$ and consider the permutation graph G_π. It can be easily checked that G_{2143} is the only permutation graph isomorphic to $K_2 \uplus K_2$ and G_{3412} is the only permutation graph isomorphic to C_4, so G_π cannot contain either of these two graphs. Moreover, G_π cannot contain C_5 because permutation graphs are perfect graphs[6], so G_π is split by

[6]Recall that the perfect graphs are those graphs G such that for all induced subgraphs H of G,

Theorem 1.6.9. Therefore the vertices of G_π can be partitioned into a clique and an independent set, and thus π can be partitioned into a decreasing subsequence and an increasing subsequence, so it is skew-merged. □

Atkinson [11] solved the enumeration problem for the skew-merged permutations; their generating function is given by

$$\frac{1-3x}{(1-2x)\sqrt{1-4x}}$$

(sequence A029759 in the OEIS [131]). We remark that this class also has a finite enumeration scheme (see Section 5.3).

Kézdy, Snevily, and Wang [86] gave a generalization of skew-merged permutations. They proved that the class of permutations which can be partitioned into r increasing subsequences and s decreasing subsequences is finitely based.

The correspondence between permutations and graphs also proves fruitful for the permutations sortable with a device known as an output-restricted deque, see Guruswami [78].

1.7.2 W-classes

The class of permutations that can be written as a union of k decreasing subsequences is precisely the class $\mathrm{Av}(12\cdots(k+1))$. What about classes that can be written as the *concatenation* of $k-1$ decreasing subsequences? Or more generally, classes that can be written as concatenations of a fixed number of decreasing and increasing subsequences? For example, the permutation 6421753 is a concatenation of two decreasing subsequences $(6,4,2,1$ and $7,5,3)$ while 1345762 can be written as the concatenation of an increasing subsequence and a decreasing subsequence in two different ways: 13457|62 or 1345|762.

---

the chromatic number of H is equal to the size of the largest clique in H. Clearly this is a closed class of graphs under the induced subgraph order, and the Strong Perfect Graph Theorem, proved by Chudnovsky, Robertson, Seymour, and Thomas [46, 47], states that the basis of this class is the set of odd cycles and their complements, $\{C_3, C_5, C_7, \ldots, \overline{C_3}, \overline{C_5}, \overline{C_7}, \ldots\}$. Despite the fact that perfect graphs are infinitely based, Chudnovsky, Cornuéjols, Liu, Seymour, and Vušković [45] give a polynomial-time algorithm to establish whether or not a given graph is perfect.

We first need to introduce notation. Fix a ± 1 row vector $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k)$. An $\mathbf{x}$-gridding of the permutation π is an expression of π as a concatenation $\pi_1\pi_2\cdots\pi_k$ where π_i is an increasing sequence if $\mathbf{x}_i = 1$ and a decreasing sequence if $\mathbf{x}_i = -1$ (empty π_i's are permitted; we consider the empty sequence to be both increasing and decreasing). The *W-class of* $\mathbf{x}$, $W(\mathbf{x})$, is then the set of all permutations with an $\mathbf{x}$-gridding. Returning to our examples in the previous paragraph we see that $6421753 \in W(-1, -1)$ and $1345762 \in W(1, -1)$. A longer example is shown in Figure 1.12. These classes have some very nice properties, as the next two theorems show.

Theorem 1.7.2 (Atkinson, Murphy, and Ruškuc [13])**.** *Every W-class is both finitely based and pwo.*

Theorem 1.7.3 (Albert, Atkinson, and Ruškuc [6])**.** *Every subclass of a W-class has a rational generating function.*

A *descent* in the permutation π is an index i for which $\pi(i) > \pi(i + 1)$. Clearly the class of permutations with at most k descents is a subclass of $W(1^{k+1})$ (where 1^{k+1} denotes the word with $k + 1$ letters all equal to 1), and thus we get the following corollary.

Corollary 1.7.4. *For any permutation class $\mathcal{C}$ and fixed integer k, the subclass of $\mathcal{C}$ consisting of all permutations with at most k descents has a rational generating function.*

Theorem 1.7.3 is a special case of the upcoming Theorem 7.1.1, while Theorem 1.7.3 would follow from Conjecture 7.1.2

1.8 Infinite antichains and partial well-order

We have already seen that several of the contexts in question — vectors in $\mathbb{N}^m$, words, and while we made no note of it, also compositions and partitions — are pwo, that is, they do not contain infinite antichains. This leaves three contexts: graphs, posets, and of course permutations, all of which do contain infinite antichains.

An infinite antichain of graphs is quite easy to spot: the cycles, $\{C_3, C_4, C_5, \ldots\}$. Typical members of two other antichains are shown in Figure 1.13.

As we remarked in Subsection 1.6.6, if $\sigma \leq \pi$ then G_σ must be an induced subgraph of G_π, so to exhibit an infinite antichain of permutations, we need only find a set of permutations that give the graphs shown in Figure 1.13 (we can't find permutations that give the cycle antichain, because permutation graphs are perfect graphs).

First we have the antichain $U = \{u_1, u_2, \dots\}$ where

$$\begin{aligned}
u_1 &= 2,3,5,1 \mid \mid 6,7,4 \\
u_2 &= 2,3,5,1 \mid 7,4 \mid 8,9,6 \\
u_3 &= 2,3,5,1 \mid 7,4,9,6 \mid 10,11,8 \\
&\vdots \\
u_k &= 2,3,5,1 \mid 7,4,9,6,11,8,\dots,2k+3,2k \mid 2k+4,2k+5,2k+2 \\
&\vdots
\end{aligned}$$

Here the vertical bars are to emphasize the different parts of the permutations, and have no mathematical meaning. The plot and permutation graph of u_7 are shown in Figure 1.14, so this is the permutation analogue of the infinite antichain of graphs from the left-hand side of Figure 1.13.

This is essentially the antichain constructed by Speilman and Bóna [134]. The idea to prove that it is an antichain from its permutation graphs is due to Klazar [88]; other proofs that it is an antichain can be found in Atkinson, Murphy, and Ruškuc [13] and Murphy's thesis [112].

Corresponding to the right-hand side of Figure 1.13 we have the antichain $T = \{t_1, t_2, \dots\}$ where

$$\begin{aligned}
t_1 &= 4,1,6,3 \mid 2,5 \\
t_2 &= 4,1,6,3,8,5 \mid 2,7 \\
t_3 &= 4,1,6,3,8,5,10,7 \mid 2,9 \\
&\vdots \\
t_k &= 4,1,6,3,8,5,10,7,\dots,2k+2,2k-1,2k+4,2k+1 \mid 2,2k+3 \\
&\vdots
\end{aligned}$$

The plot and permutation graph of t_7 are shown in Figure 1.15.

Note that the set of posets corresponding to these permutations will also form an infinite antichain of posets.

Both of these antichains are constructed from the increasing oscillating sequence discussed in Subsection 1.5.5, but there are other possibilities for antichains, for example, the $W = \{w_1, w_2, \dots\}$ is given by

$$
\begin{aligned}
w_1 &= 8, 1 \mid 5, 3, 6, 7, 9, 4 \mid \mid 10, 11, 2 \\
w_2 &= 12, 1, 10, 3 \mid 7, 5, 8, 9, 11, 6 \mid 13, 4 \mid 14, 15, 2 \\
w_3 &= 16, 1, 14, 3, 12, 5 \mid 9, 7, 10, 11, 13, 8 \mid 15, 6, 17, 4 \mid 18, 19, 2 \\
&\vdots \\
w_k &= 4k+4, 1, 4k+2, 3, \dots, 2k+6, 2k-1 \mid \\
&\quad 2k+3, 2k+1, 2k+4, 2k+5, 2k+7, 2k+2 \mid \\
&\quad 2k+9, 2k, 2k+11, 2k-2, \dots, 4k+5, 4 \mid \\
&\quad 4k+6, 4k+7, 2 \\
&\vdots
\end{aligned}
$$

where the vertical bars indicate that w_k consists of four different parts, of which the first part is the interleaving of $4k+4, 4k+2, \dots, 2k+6$ with $1, 3, \dots, 2k-1$, the second part consists of just six terms, the third part is the interleaving of $2k+9, 2k+11, \dots, 4k+5$ with $2k, 2k-2, \dots, 4$, and the fourth part has three terms. The plot of w_6 is shown in Figure 1.16. Atkinson, Murphy and Ruškuc [13] give a proof that W is an antichain. This fact, which does not seem to have a completely trivial proof like for U and T, is a special case of Theorem 7.1.1.

Chapter 2

A review of general enumerative results and open problems

2.1 The Marcus-Tardos proof of the Füredi-Hajnal and Stanley-Wilf Conjectures

In the early 1990's Stanley and Wilf conjectured that all permutation classes except the class of all permutations have at most exponential counting functions.

Marcus-Tardos Theorem A (formerly the Stanley-Wilf Conjecture). *If the permutation class $\mathcal{C}$ is not the set of all permutations then there is a number K depending only on $\mathcal{C}$ so that $s_n(\mathcal{C}) < K^n$ for all n.*

Despite the efforts of many researchers, this conjecture remained open for more than a decade until its recent resolution by Marcus and Tardos [105]. Two noteworthy partial results appeared during this decade. First Bóna [32] proved by a complicated argument that the Stanley-Wilf Conjecture holds whenever $\mathcal{C}$ avoids a layered permutation. Then Alon and Friedgut [8] used results about Davenport-Schinzel sequences to prove that $s_n(\mathcal{C}) < K^{n\gamma(n)}$ where γ is an extremely slowly growing function. This result caused Wilf's belief in the conjecture to waver; he wrote in [161] that

> This conjecture had been considered a "sure thing," but the results of Alon and Friedgut seem to make it somewhat less certain because a similar bound, involving the Ackermann function, in the Davenport-Schinzel theory turns out to be best possible.

Then Marcus and Tardos [105] presented an extremely elegant and elementary proof that proves not only the Stanley-Wilf Conjecture but also a newer conjecture of Füredi and Hajnal [73]:

Marcus-Tardos Theorem B (formerly the Füredi-Hajnal Conjecture). *For all permutations* β*,* $\mathrm{ex}(n, M^\beta) = O(n)$*.*

About their proof, Zeilberger [163] wrote

> Just because a conjecture has been posed by brilliant people, and attempted, unsuccessfully, by quite a few other brilliant people, does not mean that the ultimate proof has to be complicated. We saw this a year ago with the AKS Primality is P result, and now the brilliant proof of the Füredi-Hajnal conjecture, and hence the Stanley-Wilf Conjecture by "0-th year grad student" Adam Marcus (now at GA Tech) and Gabor Tardos. Once I saw their proof, I kicked myself, as I am sure did many other people. Once you see it is so NATURAL and "obvious." But if it is so "obvious," how come no one (including myself) did come up with it? So like most of the truly great discoveries, it is a posteriori "obvious," but definitely not "a priori."

2.1.1 The Füredi-Hajnal Conjecture implies the Stanley-Wilf Conjecture

We begin by giving Klazar's proof [87] that the Füredi-Hajnal Conjecture implies the Stanley-Wilf Conjecture. Let β be a permutation and temporarily assume the truth of the Füredi-Hajnal Conjecture, so $\mathrm{ex}(n, M^\beta) \leq Kn$ for some constant K. We will show here that

$$s_{n\times n}(M^\beta) \leq 15^{2Kn}. \tag{2.1}$$

This will prove the Stanley-Wilf Conjecture because all permutation classes that do not contain every permutation lie in $\mathrm{Av}(\beta)$ for some β, and if π is a β-avoiding permutation then M^π avoids M^β.

In order to prove (2.1), take some $M \in \mathrm{Av}_{n\times n}(M^\beta)$. We will assume for the time being that n is a multiple of 2. The crucial idea in Klazar's proof, a variant of which is also the crucial idea of the Marcus-Tardos proof, is to condense M into a matrix in $\mathrm{Av}_{n/2\times n/2}(M^\beta)$. We do this by applying the following "combinatorial wavelet transformation."

Divide M into 2×2 blocks, and define the $n/2\times n/2$ matrix N by

$$N_{i,j} = \begin{cases} 1 & \text{if } M_{[2i-1,2i]\times[2j-1,2j]} \text{ contains a 1,} \\ 0 & \text{if } M_{[2i-1,2i]\times[2j-1,2j]} \text{ is the 0-matrix.} \end{cases}$$

Lemma 2.1.1. *N avoids M^β.*

Proof. If N were to contain M^β then, since M^β is a permutation matrix, one could reconstruct a copy of M^β in M. □

Now consider a specific $N\in \mathrm{Av}_{n/2\times n/2}(M^\beta)$. If N has m 1's, then it could come from at most 15^m matrices from $\mathrm{Av}_{n\times n}(M^\beta)$ because for each 1 in N there are only 15 nonzero 2×2 blocks that M could have in that position. Furthermore, we are assuming that N has at most $Kn/2$ 1's (because it avoids M^β). Thus we have shown that

$$s_{n\times n}(M^\beta) \le 15^{Kn/2} s_{n/2\times n/2}(M^\beta). \tag{2.2}$$

When n is odd, we can combine (2.2) with the inequality $s_{n\times n}(M^\beta)\le s_{(n+1)\times(n+1)}(M^\beta)$ to get a bound. Applying these bounds repeatedly until $s_{1\times 1}(M^\beta)\le 2$ is reached, we get that

$$s_{n\times n}(M^\beta) \le 15^{K\lceil n/2\rceil}\cdot 15^{K\lceil n/4\rceil}\cdot 15^{K\lceil n/8\rceil}\cdots 2 \le 15^{2Kn}, \tag{2.3}$$

proving (2.1). All we have left is then to prove the Füredi-Hajnal Conjecture.

2.1.2 The proof of the Füredi-Hajnal Conjecture

Take a permutation $\beta\in S_k$ and suppose that M avoids M^β and has the maximum possible number of 1's, $\mathrm{ex}(n, M^\beta)$. Again we condense M^β into a smaller matrix, although this time we condense much more. We will leave this condensation factor as a variable for now, say a, and will suppose (harmlessly) that it divides n. Now divide M into $a\times a$ blocks, defining

$$S_{i,j} = M_{[(i-1)a+1,ia]\times[(j-1)a+1,ja]},$$

for $i, j \in [n/a]$ and set

$$N_{i,j} = \begin{cases} 1 & \text{if } S_{i,j} \text{ contains a 1,} \\ 0 & \text{if } S_{i,j} \text{ is the 0-matrix.} \end{cases}$$

Again, observe that N avoids M^β. As with Klazar's proof, we also know that N has at most $\mathrm{ex}(n/a, M^\beta)$ entries equal to 1, but unlike Klazar's proof, we need to do some more work.

The key to the proof is the following definition. A block $S_{i,j}$ is said to be *wide* if it contains 1's in at least k different columns (recall that k is the length of β), or *tall* if it contains 1's in at least k different rows.

Lemma 2.1.2. *For any i, there are less than $k\binom{a}{k}$ tall blocks $S_{i,j}$.*

Proof. If not, then by the pigeonhole principle, there are k blocks that have 1's in the same set of k rows, and from this one is able to find a copy of M^β for every $\beta \in S_k$. □

Completely analogously, for any j, less than $k\binom{a}{k}$ blocks $S_{i,j}$ can be wide. Now we have all we need to get a linear upper bound. There are n/a rows of blocks in M, and at most $k\binom{a}{k}$ blocks in any row can be tall, so there are no more than $\frac{nk}{a}\binom{a}{k}$ tall blocks in total, and similarly no more than $\frac{nk}{a}\binom{a}{k}$ wide blocks. Each of these have at most a^2 entries equal to 1. Of the other blocks, we know that no more than $\mathrm{ex}(n/a, M^\beta)$ of them can be nonzero because N avoids M^β, and these nonzero blocks can have at most $(k-1)^2$ 1's, because they are neither tall nor wide. Therefore we have

$$\mathrm{ex}(n, M^\beta) \le 2\frac{nk}{a}\binom{a}{k} + (k-1)^2\,\mathrm{ex}(n/a, M^\beta). \tag{2.4}$$

Setting $a = k^2$ and solving (2.4) gives

$$\mathrm{ex}(n, M^\beta) \le 2k^4\binom{k^2}{k}n,$$

proving the Füredi-Hajnal Conjecture. Now by applying the bounds from the previous subsection we have:

Corollary 2.1.3. *For any permutation $\beta \in S_k$,*

$$s_n(\beta) \le \left(15^{4k^4\binom{k^2}{k}}\right)^n.$$

This resolves one of the questions raised by Füredi and Hajnal, but several others remain open, for example, the following.

Problem 2.1.4 (Füredi and Hajnal [73]). *Characterize all matrices M such that* $\mathrm{ex}(n, M) = O(n)$*.*

It is clear that the set of matrices with linear extremal functions asked for above is a closed class including all permutation matrices. It also includes non-permutation matrices, as Füredi and Hajnal observed, and more examples were found by Tardos [149].

2.2 Wilf-equivalence

The classes $\mathcal{C}_1$ and $\mathcal{C}_2$ are said to be *Wilf-equivalent* if they are equinumerous, that is, if $s_n(\mathcal{C}_1) = s_n(\mathcal{C}_2)$ for all natural numbers n. Obviously the symmetries of a class are Wilf-equivalent to the class, but many more examples of non-trivial Wilf-equivalences have been observed, ranging from the fact every class defined by avoiding a single permutation of length three is Wilf-equivalent (see Subsection 1.4.1) to the theorem of Atkinson, Murphy, and Ruškuc [14] that Av(1342) is Wilf-equivalent to the infinitely based class

$$\mathrm{Av}(\{2(2m-1)416385\cdots(2m)(2m-3) : m = 2, 3, \ldots\}).$$

There are also bases of different finite cardinalities that give rise to Wilf-equivalent classes. Infinitely many examples of this form can be computed via Corollaries 3.7 and 3.9 of Vatter [156].

We have two general results about the Wilf-equivalence of singleton-based classes.

Theorem 2.2.1 (Backelin, West, and Xin [18]). *For any k and permutation β, the classes* $\mathrm{Av}(12\cdots k \oplus \beta)$ *and* $\mathrm{Av}(k\cdots 21 \oplus \beta)$ *are Wilf-equivalent.*

Theorem 2.2.2 (Stankova and West [136]). *For all permutations β, the classes* $\mathrm{Av}(213 \oplus \beta)$ *and* $\mathrm{Av}(312 \oplus \beta)$ *are Wilf-equivalent.*

The case $k = 2$ of Theorem 2.2.1 was proved by West in his thesis [157], while the $k = 3$ case was proved by Babson and West [17]. The full version of the theorem has since been proved in the context of involutions by Bousquet-Mélou and Steingrímsson [39].

Regarding the Wilf-equivalence of singleton-based classes, there is only one more result known. It is the following sporadic equivalence.

Theorem 2.2.3 (Stankova [138])**.** *The classes* Av(4132) *and* Av(3142) *are Wilf-equivalent.*

Theorems 2.2.1, 2.2.2, and 2.2.3, together with the trivial symmetries, characterize all Wilf-equivalences between classes of the form $\text{Av}(\beta)$ with $|\beta| \leq 7$. Given the success of this combination of results, we may well be close to an answer to the following.

Problem 2.2.4. *Characterize all Wilf-equivalences among singleton based classes.*

Three other characterizations have been provided to-date: Le [94] characterizes the Wilf-equivalences among classes avoiding two permutations of length 4, Lipson [95] settles the problem for classes avoiding a permutation of length 3 and another of length 5, and Theorem 1.6.8 solves the problem for classes of the form $\text{Av}(231, 312, \beta)$.

While Problem 2.2.4 asks for a complete characterization of Wilf-equivalence of singleton-based sets, in general a decidability result may be the best that could be hoped for.

Question 2.2.5. *Is it decidable whether two finitely based classes are Wilf-equivalent?*

2.3 Containing a permutation at least once

While Theorems 2.2.1 and 2.2.2 provide powerful sufficient conditions for the Wilf-equivalence of singleton-based classes, necessary conditions are almost completely missing. We describe one possible approach to developing necessary conditions here.

Let $\pi \in S_m$ be any permutation. For a positive integer k, define $t_k(\pi)$ to be the number of permutations of length $m + k$ that *contain* π, so

$$t_k(\pi) = (m + k)! - s_{m+k}(\pi).$$

This may appear on first glance to be nothing more than a restatement of the enumeration problem, but it is not. In the enumeration problem we think of π as fixed and k as varying. Here we will think of k as fixed and π as varying. What can be said of $t_k(\pi)$?

It turns out that $t_1(\pi)$ is not difficult to find. Pratt [120] stated this result but left its proof to the reader[1], while Ray and West [123] present a proof due to Bloomberg.

Proposition 2.3.1 (Pratt [120]; Bloomberg [unpublished, 1990], see Ray and West [123]). *For any permutation $\pi \in S_m$ we have $t_1(\pi) = m^2 + 1$.*

Ray and West [123] found that $t_2(\pi)$ depends on π:

Theorem 2.3.2 (Ray and West [123]). *For any permutation $\pi \in S_m$ we have*

$$t_2(\pi) = (m^4 + 2m^3 + m^2 + 4m + 4 - 2j)/2$$

where $0 \leq j \leq m - 1$ depends on π but not on m.

The next value of k remains unexplored.

Question 2.3.3. *What is $t_3(\pi)$?*

Ray and West also proved a result giving the largest-order terms of t_k.

Theorem 2.3.4 (Ray and West [123]). *For any permutation $\pi \in S_m$ we have*

$$t_k(\pi) = (m^{2k} + k(k-1)m^{2k-1})/k! + O(m^{2k-2}).$$

Furthermore, Ray and West showed using the Robinson-Schensted-Knuth correspondence that $t_k(12 \cdots m)$ is a polynomial in m.

2.4 How do they compare?

Many had thought that the permutations of a certain length could be totally ordered according to avoidance modulo Wilf-equivalence; to be more precise, it was thought that the permutations in S_k could be labeled $\beta_1, \beta_2, \ldots, \beta_{k!}$ so that for all n, $s_n(\beta_i) \leq s_n(\beta_j)$ whenever $i \leq j$. Stankova and West [136] were the first to find a counterexample. They observed that $s_n(53241) < s_n(43251)$ for all $n \leq 12$ while $s_{13}(53241) > s_{13}(43251)$.

Stankova and West [136] conjectured that the permutations of a certain length could be asymptotically ordered according to avoidance, modulo Wilf-equivalence, where we

[1]The reasons Pratt did this will become obvious to the reader who attempts to write a careful solution.

say that β_2 is asymptotically more avoidable that β_1 if $s_n(\beta_2) > s_n(\beta_1)$ for all sufficiently large n. This conjecture has yet to be disproved. Burstein presented several more specific conjectures about the comparison of singleton-based classes at the Third Annual International Conference on Permutation Patterns, see Elder and Vatter [60].

Also presented at that conference was a conjecture of Bóna. First recall the permutation posets of Subsection 1.6.5; our two examples are repeated in Figure 2.1.

Permutation posets are ranked, where the rank of i in P_π is the length of the longest increasing subsequence of π that ends in i. Given a poset P with rank function r, we let $conv(P)$ denote the *convex hull* of P. The poset $conv(P)$ is defined on the same set of elements as P, but in $conv(P)$, $i \preceq_{conv} j$ if and only if $r(i) \leq r(j)$ (in P). Figure 2.2 shows the convex hulls of the posets from Figure 2.1. Note that $conv(P_{215436}) = P_{215436}$ and that $conv(P_{416352}) \cong P_{215436}$. Indeed, these are not accidents. For any layered permutation π, $conv(P_\pi) = P$, and for any permutation π there is a unique layered permutation, which we refer to as $conv(\pi)$, such that $P_\pi \cong P_{conv(\pi)}$.

Conjecture 2.4.1 (Bóna, see Elder and Vatter [60]). *For any permutation π and natural number n, $s_n(\pi) \leq s_n(conv(\pi))$.*

Via Bóna [32], an affirmative resolution of Conjecture 2.4.1 would give a new proof of the Stanley-Wilf Conjecture.

2.5 Do they have nice formulas? (The Gessel-Noonan-Zeilberger Conjecture)

A generating function is said to be *holonomic* (or synonymously, in the univariate case, *D-finite*) if its derivatives span a finite dimensional subspace over $\mathbb{C}(x)$. This is equivalent to the corresponding sequence s_n being holonomic (again synonymously in the univariate case, *P-recursive*), which means that there are polynomials $p_0, p_1, \ldots p_k$ so that

$$p_k(n)s_{n+k} + p_{k-1}(n)s_{n+k-1} + \cdots + p_0(n)s_n = 0.$$

It is known that every algebraic generating function is also holonomic.

We have already mentioned that Av(1234) has a non-algebraic generating function, so the next best thing we could hope for is if every permutation class has a holonomic generating function. However, this immediately turns out to be too much to hope for.

Proposition 2.5.1 (Atkinson and Stitt [16]; Murphy [112, Chapter 9]). *If the permutation class $\mathcal{C}$ contains an infinite antichain then it also contains a subclass with a non-holonomic generating function. In particular, there are permutation classes with non-holonomic generating functions.*

Proof. Choose an infinite antichain $A \subseteq \mathcal{C}$ that has at most one element of each length. If $A_1 \neq A_2$ are two subsets of A then the two subclasses $\mathcal{C} \cap \mathrm{Av}(A_1)$ and $\mathcal{C} \cap \mathrm{Av}(A_2)$ have different enumerations. Because A is infinite, this gives $2^{\aleph_0}$ different generating functions. Now notice that if f is a holonomic generating function for a permutation class then the recurrence satisfied by f may be chosen to have integral coefficients and integral initial conditions, and so there are only countably many holonomic generating functions for permutation classes. □

Notice, however, that Proposition 2.5.1 does not preclude all finitely based permutation classes from having holonomic generating functions (keeping the notation from that proof, there are only countably many subclasses of the form $\mathcal{C} \cap \mathrm{Av}(A_1)$ where A_1 is finite). Noonan and Zeilberger [116] conjectured that all such classes have holonomic generating functions, while Gessel [74] was more cautious, writing that

> Another possible candidate for P-recursiveness is the problem of counting permutations (or more generally sequences) with forbidden subsequences defined by inequalities, for example permutations $a_1 a_2 \cdots a_n$ of $\{1, 2, \ldots, n\}$ with no subsequence $a_i a_j a_k$ satisfying $a_i < a_k < a_j$.

Gessel-Noonan-Zeilberger Conjecture. *All finitely based permutation classes have holonomic generating functions.*

In fact, Noonan and Zeilberger made what seemed to be a stronger conjecture. Let $\beta_1, \beta_2, \ldots, \beta_k$ be permutations and $r_1, r_2, \ldots, r_k$ nonnegative integers. Noonan and

Zeilberger conjectured that the number of n-permutations with at most[2] r_i copies of β_i for each $i \in [k]$ forms a holonomic sequence in n. First note that the set of these permutations forms a permutation class; we will denote this class by

$$\mathrm{Av}(\beta_1^{\leq r_1}, \beta_2^{\leq r_2}, \ldots, \beta_k^{\leq r_k}).$$

Atkinson observed that this seemingly stronger conjecture is equivalent to the Gessel-Noonan-Zeilberger Conjecture. This equivalence follows from the following result.

Proposition 2.5.2 (Atkinson [12])**.** *For any choice of permutations $\beta_1, \beta_2, \ldots, \beta_k$ and nonnegative integers $r_1, r_2, \ldots, r_k$, the class $\mathrm{Av}(\beta_1^{\leq r_1}, \beta_2^{\leq r_2}, \ldots, \beta_k^{\leq r_k})$ is finitely based.*

Proof. Let π denote a basis element of $\mathcal{C} = \mathrm{Av}(\beta_1^{\leq r_1}, \beta_2^{\leq r_2}, \ldots, \beta_k^{\leq r_k})$, so for some $i \in [k]$, π has more than r_i copies of β_i. Now choose $r_i + 1$ copies of β_i from π and construct the permutation $\sigma \leq \pi$ by taking every element contained in a chosen copy of β_i and standardizing. By construction $|\sigma| \leq (r_i + 1)|\beta_i|$, and clearly $\sigma \leq \pi$ and $\sigma \notin \mathcal{C}$, so we must have $\sigma = \pi$ by the minimality of π. Thus the basis elements of $\mathcal{C}$ are all of bounded length, so there can only be finitely many of them. $\square$

For readers interested in computations, the basis of $\mathrm{Av}(123^{\leq 1})$ is

$$\{1234, 1243, 1324, 2134, 14523, 34125, 351624, 356124, 451623, 456123\}$$

(this class was first counted by Noonan [115]), and the basis of $\mathrm{Av}(132^{\leq 1})$ is

$$\{1243, 1342, 1423, 1432, 2143, 35142, 354162, 461325, 465132\}$$

Later Fulmek [72] counted $\mathrm{Av}(123^{\leq 2})$. The most general result for a class of the form $\mathrm{Av}(\beta^{\leq r})$ is the following.

Theorem 2.5.3 (Bóna [31]; Mansour and Vainshtein [104])**.** *For any fixed r, the class $\mathrm{Av}(132^{\leq r})$ has an algebraic generating function.*

---

[2]This is a bit of a lie; to get their precise conjecture, replace "at most" by "exactly," but by inclusion-exclusion the two versions are equivalent.

As remarked in Subsection 1.4.3, we now know the first 26 terms of the sequence $s_n(1324)$. One might hope to be able to guess a polynomial recurrence for $s_n(1324)$ from this data, but no such recurrence fits. Moreover, the data suggests that $s_n(1324)$ is not (asymptotically) of the form $n^\alpha C^n$, as it should be if it is a holonomic sequence. Disheartened by this, Zeilberger recently conjectured that the Gessel-Noonan-Zeilberger Conjecture is false.

Conjecture 2.5.4 (Zeilberger, see Elder and Vatter [60])**.** *The Gessel-Noonan-Zeilberger Conjecture is false, and, in particular, the sequence* $s_n(1324)$ *is not holonomic.*

If the sequence were holonomic, then there would be a polynomial time algorithm (the recurrence) to compute $s_n(1324)$. Zeilberger went further to conjecture that there is no such algorithm, and that "not even God knows $s_{1000}(1324)$."

2.6 Growth rates

In order to get a rough measure of how large the permutation class $\mathcal{C}$ is, we would like to define the *growth rate* of $\mathcal{C}$ as

$$\mathrm{gr}(\mathcal{C}) = \lim_{n\to\infty} \sqrt[n]{s_n(\mathcal{C})}.$$

Unfortunately, this limit is not known to exist for all classes $\mathcal{C}$, or even all finitely based classes, or even all doubleton-based classes. But there doesn't seem to be any reason to doubt the following conjecture.

Conjecture 2.6.1. *For all permutation classes* $\mathcal{C}$*,* $\lim_{n\to\infty} \sqrt[n]{s_n(\mathcal{C})}$ *exists.*

We do have one special case of Conjecture 2.6.1; it holds for singleton-based classes. We present the short proof below.

Proposition 2.6.2 (Arratia [10])**.** *For every permutation* β*, the growth rate of* $\mathrm{Av}(\beta)$*,* $\lim_{n\to\infty} \sqrt[n]{s_n(\beta)}$*, exists.*

Proof. We remarked in Section 1.5 that every singeton-based class is either $\oplus$-complete or $\ominus$-complete, so suppose without loss that $\mathrm{Av}(\beta)$ is $\oplus$-complete. Then $\oplus$ gives an injection from $\mathrm{Av}_m(\beta) \times \mathrm{Av}_n(\beta)$ to $\mathrm{Av}_{m+n}(\beta)$, so the sequence $s_n(\beta)$ is supermultiplicative and thus the specified limit exists by Fekete's Lemma[3]. □

Stanley had conjectured that $\mathrm{gr}(\mathrm{Av}(\beta)) = (k-1)^2$ for all $\beta \in S_k$. This was disproved by Bóna [30], who showed that $\mathrm{gr}(1342) = 8$. The next natural conjecture would be that $\mathrm{gr}(\mathrm{Av}(\beta))$ is always an integer. This too was disproved by Bóna:

Theorem 2.6.3 (Bóna [35]). *For any* $k \geq 1$, $\mathrm{gr}(\mathrm{Av}(12\cdots k \oplus 231)) = (k-4+\sqrt{8})^2$.

Inspired by this result, Bóna went on to ask the following.

Question 2.6.4 (Bóna [35]). *Are growth rates of singleton-based classes necessarily algebraic? Are they algebraic of degree two? Are they always algebraic integers?*

Corollary 2.1.3 shows that $\mathrm{gr}(\mathrm{Av}(\beta)) \leq 15^{4k^4\binom{k^2}{k}}$ for all $\beta \in S_k$, but this is commonly thought to be a huge overestimate.

Question 2.6.5. *Define*

$$f(k) = \max_{\beta \in S_k} \mathrm{gr}(\mathrm{Av}(\beta)).$$

Does $f(k) = O(k^2)$? *Is* $f(k)$ *at least sub-exponential?*

As mentioned in Section 1.4, Arratia [10] had conjectured that $f(k) = (k-1)^2$, but this was disproved by Albert, Elder, Rechnitzer, Westcott and Zabrocki [1].

One could also ask which β's give $f(k)$. Bóna has a conjecture on this.

Conjecture 2.6.6 (Bóna [29]). *For odd* k, $f(k)$ *is the growth rate of the* $1 \oplus 21 \oplus 21 \oplus \cdots \oplus 21$*-avoiding permutations, while for even* k *it is the growth rate of the* $1 \oplus 21 \oplus 21 \oplus \cdots \oplus 21 \oplus 1$*-avoiding permutations.*

2.6.1 The smallest non-pwo class?

Because growth rates are not known to exist in general, for non-singleton-based classes we will instead take the lim sup and define the *upper growth rate* of the permutation

---

[3]Fekete's Lemma in its usual form says that if a a_n is superadditive, meaning that $a_{m+n} \geq a_m + a_n$, then $\lim_{n\to\infty} a_n/n$ exists and is equal to $\sup a_n/n$. To apply this form of Fekete's Lemma in our context, consider the sequence $\log s_n(\beta)$.

class $\mathcal{C}$ as

$$\overline{\mathrm{gr}}(\mathcal{C}) = \limsup_{n\to\infty} \sqrt[n]{s_n(\mathcal{C})}.$$

Note that by the Marcus-Tardos Theorem, $\overline{\mathrm{gr}}(\mathcal{C}) < \infty$ for all classes $\mathcal{C}$ except the class containing all permutations.

Klazar [88] proves that all classes $\mathcal{C}$ with $\overline{\mathrm{gr}}(\mathcal{C}) < 2$ are pwo, so "small enough" classes must be pwo. In the other direction, what is the smallest non-pwo class[4]? Let us define

$$\psi = \inf\{\overline{\mathrm{gr}}(\mathcal{C}) : \text{permutation classes } \mathcal{C} \text{ containing an infinite antichain}\}.$$

The antichain U from Section 1.8 lies in Av(321), so since $s_n(321)$ in the nth Catalan number, $\overline{\mathrm{gr}}(\mathrm{Av}(321)) = 4$, and thus $\psi \leq 4$. Indeed, it can be shown that

$$U \subseteq \mathrm{Av}(321, 4123, 3412, 23451, 314526, 314625, 134526, 134625), \tag{2.5}$$

and FinLabel computes the generating function for this class to be

$$\frac{x(1 + x + x^2 + 2x^3 + 3x^4 + 3x^5 + x^6 - x^7 - x^9)}{(1 + x)(1 - 2x - x^3)}, \tag{2.6}$$

from which it follows that

$$\psi \leq \text{the unique real root of } 1 + 2x^2 - x^3 \approx 2.20556.$$

Murphy and I conjectured that ψ is precisely this constant:

Conjecture 2.6.7 (Murphy and Vatter, see Klazar [88])**.** *ψ is the unique real root of $1 + 2x^2 - x^3$.*

The motivation for these investigations was a question Klazar [88] raised. For any positive real number α, let

$$K_\alpha = \{\text{permutation classes } \mathcal{C} : \overline{\mathrm{gr}}(\mathcal{C}) < \alpha\}.$$

First we observe the connection between the cardinality of K_α and non-pwo classes.

[4]Strictly speaking there is no "smallest non-pwo class," because any non-pwo class contains an infinite descending chain of non-pwo classes.

Proposition 2.6.8. *Suppose the permutation class $\mathcal{C}$ contains an infinite antichain and that $\overline{\mathrm{gr}}(\mathcal{C}) < \alpha$. Then K_α is uncountable.*

Proof. Let Y denote an infinite antichain in $\mathcal{C}$. Then the set of permutation classes $\{\mathcal{C} \cap \mathrm{Av}(X) : X \subseteq Y\}$ is uncountable, and since these classes have upper growth rate at most $\overline{\mathrm{gr}}(\mathcal{C}) < \alpha$, K_α is uncountable. ☐

Because the class from (2.5) contains an infinite antichain, K_α is uncountable for all α greater than the unique real root of $1 + 2x^2 - x^3$. Klazar asked about the "phase transition" from countability to uncountability, or in other words, the constant

$$\kappa = \inf\{\alpha : \text{the set } K_\alpha \text{ is uncountable}\}.$$

Proposition 2.6.8 implies that $\kappa \leq \psi$, but the other direction remains open.

Conjecture 2.6.9. $\kappa = \psi$.

Chapter 3

Simplicity, atomicity, and unimodality for permutations and, more generally, relational structures

3.1 Relational structures

The partial orders we have chosen to discuss are all intimately connected by the fact that they can be viewed as relational structures. Here we discuss this more general context, which appears to be the proper level of generality for several results. In addition, while there are differing interpretations of what "infinite permutation" ought to mean, the relational structure viewpoint we will adopt suggests a natural interpretation for this term, which we discuss in Section 3.3.

A *k-ary relation* R on the ground set A is a subset of A^k. A *relational structure* (or *relational system*) over the ground set A is then an ordered sequence of relations over A. However, in order to define containment and isomorphism precisely we introduce some new notation.

We will say that a *relational language* (or *relational signature*), $\mathcal{L}$, is a set of symbols R (called *relational symbols*) together with positive integers n_R for each $R \in \mathcal{L}$. A relational language is said to be *finite* if it contains only finitely many relational symbols. An $\mathcal{L}$-structure $\mathcal{A}$, which we will also call a *relational structure*, consists of a ground set $\mathrm{dom}(\mathcal{A})$ and a set of subsets $R^{\mathcal{A}} \subseteq \mathrm{dom}(\mathcal{A})^{n_R}$ for each $R \in \mathcal{L}$. If $(a_1, a_2, \ldots, a_{n_R}) \in R^{\mathcal{A}}$ we write $R^{\mathcal{A}}(a_1, a_2, \ldots, a_{n_R})$. Thus $R^{\mathcal{A}}$ is an n_R-ary relation; it is also called an *interpretation* of R.

Let $\mathcal{L}$ be a relational language and $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{L}$-structures. An *embedding* φ of $\mathcal{A}$ into $\mathcal{B}$ is an injective function from $\mathrm{dom}(\mathcal{A})$ to $\mathrm{dom}(\mathcal{B})$ such $R^{\mathcal{A}}(a_1, a_2, \ldots, a_{n_R})$ if and only if $R^{\mathcal{B}}(\varphi(a_1), \varphi(a_2), \ldots, \varphi(a_{n_R}))$ for all $R \in \mathcal{L}$ and all $a_1, a_2, \ldots, a_{n_R} \in \mathrm{dom}(\mathcal{A})$. An *isomorphism* between $\mathcal{A}$ and $\mathcal{B}$ is a surjective embedding. If there is embedding of $\mathcal{A}$ into $\mathcal{B}$ we write $\mathcal{A} \leq \mathcal{B}$. As with the induced subgraph order on graphs, this is merely a quasi-order, so we will often discuss isomorphism classes of relational structures.

3.1.1 Graphs

The correspondence between graphs and relational structures is quite clear, since a graph is by definition a set of vertices $V(G)$ together with a binary symmetric relation, E. So the relational language is $\mathcal{L} = \{E\}$ where $n_E = 2$, and the graph G corresponds to the $\mathcal{L}$-structure $\mathcal{A}_G$ where $\mathrm{dom}(\mathcal{A}_G) = V(G)$ and $E^{\mathcal{A}_G}(x, y)$ if and only if $x \sim y$ in G.

3.1.2 Posets

A poset is a relation, and thereby a relational structure, by definition.

3.1.3 Words

The connection between words and relational structures is a little less clear. Words over A can be captured with the relational language $\mathcal{L} = \{<\} \cup \{R_a : a \in A\}$ where $<$ is a binary relation and each R_a is unary. Then the word $w = w(1)w(2)\cdots w(n) \in A^*$ corresponds to the $\mathcal{L}$-structure $\mathcal{A}_w$ with ground set $[n]$, where $<^{\mathcal{A}_w}$ is interpreted as the normal integer order on $[n]$ and $R_a^{\mathcal{A}_w}(i) \iff w(i) = a$. Upon the unraveling of this definition, the following proposition becomes clear.

Proposition 3.1.1. *Let u and v be words in A^*. Then $u \leq v$ in the subword order if and only $\mathcal{A}_u$ embeds into $\mathcal{A}_v$.*

Proof. Let $u = u(1)u(2)\cdots u(k)$ and $w = w(1)w(2)\cdots w(n)$. First suppose that $u \leq w$, so there are indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ so that $w(i_1)w(i_2)\cdots w(i_k) = u$. Then the map $\varphi : [k] \to \{i_1, i_2, \ldots, i_k\}$ given by $j \mapsto i_j$ is an embedding of $\mathcal{A}_u$ into $\mathcal{A}_v$.

Now suppose that $\mathcal{A}_u \leq \mathcal{A}_v$. Then there is some embedding $\varphi : [k] \to [n]$. Label the image of this embedding $\{i_1 <^{\mathcal{A}_v} i_2 <^{\mathcal{A}_v} \cdots <^{\mathcal{A}_v} i_k\}$. Since $j <^{\mathcal{A}_u} \ell \iff \varphi(j) <^{\mathcal{A}_v}$

$\varphi(\ell)$, φ must map j to i_j. Therefore, since we also have $R_a^{\mathcal{A}_u}(j) \iff R_a^{\mathcal{A}_v}(\varphi(j)) \iff R_a^{\mathcal{A}_v}(i_j)$, we get that $u = w(i_1)w(i_2)\cdots w(i_k)$, completing the proof. □

3.1.4 Permutations

The relational language encapsulating permutations is that of two binary relations, $\mathcal{L} = \{<, \prec\}$. The permutation $\pi \in S_n$ then corresponds to the $\mathcal{L}$-structure $\mathcal{A}_\pi$ on the ground set $[n]$ where $<^{\mathcal{A}_\pi}$ is the normal ordering of $[n]$ and $i \prec^{\mathcal{A}_\pi} j \iff \pi(i) < \pi(j)$. It is a reassuring fact that the containment order we have been studying on permutations, which can seem a bit contrived when presented as it was in Section 1.1, corresponds precisely to the substructure order on $\mathcal{L}$-structures.

3.1.5 Compositions

Since the composition poset we are considering is isomorphic to the poset of layered permutations, we can use the correspondence between permutations and relational structures for this case.

3.1.6 Partitions

The partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ corresponds to a single equivalence relation on n elements with k equivalence classes of size $\lambda_1, \lambda_2, \ldots, \lambda_k$.

3.2 Intervals & simplicity

Substitution decompositions (also known as modular decompositions, disjunctive decompositions, and X-joins) have proved to be a useful technique in a wide range of settings, ranging from game theory to combinatorial optimization (see Möhring [110] or Möhring and Radermacher [111] for extensive references). Although substitution decompositions are most often applied to algorithmic problems, we will instead discuss the enumerative and structural applications of substitution decompositions, and, especially, the objects which cannot be decomposed and are therefore termed *simple*.

We begin with permutations. An *interval* (also called a *block* or *segment*, or in other contexts, *factor*, *module*, *clan*, *congruence*, or *convex subset*) in the permutation π is an interval of indices $I = [a, b]$ such that the set of values $\pi(I) = \{\pi(i) : i \in I\}$ also forms an interval of natural numbers. Clearly every permutation of length n has many *trivial intervals*: the empty set, the n intervals of length one, and the interval of length n. A permutation that has no nontrivial intervals is called *simple* (the analogous term in other contexts is often *indecomposable*, *prime*, or *primitive*). Figure 3.1 shows the plot of a permutation with its nontrivial intervals indicated.

Now let $\mathcal{L}$ denote a relational language and $\mathcal{A}$ an $\mathcal{L}$-structure. We say that the subset $X \subseteq \text{dom}(\mathcal{A})$ is an interval of $\mathcal{A}$ if for every relation $R \in \mathcal{L}$ and every n_R-tuple $(x_1, x_2, \ldots, x_{n_R}) \in \text{dom}(\mathcal{A})^{n_R} \setminus X^{n_R}$, if $x_i \in X$ then the value of $R^{\mathcal{A}}(x_1, x_2, \ldots, x_{n_R})$ is unchanged by swapping x_i with any other element of X.

The study of intervals in general relational structures began with Fraïssé [69, 70, 71]. While Fraïssé's definition of intervals in similar to ours, it is not equivalent; the definition given here was introduced under the name *partie solidaire* by Földes and Hammer [67]. Földes [65] later called them *strong intervals*.

Under this definition the nontrivial intervals in an equivalence relation are sets of blocks of the equivalence relation and subsets of a single block, while intervals in our relational structure interpretation of words are simply sets of consecutive letters. Graphs are more interesting; an interval in the graph G is a subset $X \subseteq V(G)$ so that $N(x) \setminus X = N(y) \setminus X$ for all $x, y \in X$ where $N(x)$ denotes the *open neighborhood* of the vertex x, the set $\{y : x \sim y\}$. In particular, every connected component of a graph forms an interval.

As with permutations, a relational structure $\mathcal{A}$ will have the trivial intervals $\emptyset$, $\{a\}$ for all $a \in \text{dom}(\mathcal{A})$, and $\text{dom}(A)$ itself. If the structure has no other intervals, then it is said to be *simple*. Three examples of simple permutations are shown in Figure 3.2.

3.2.1 Simple retracts

Suppose we are given a simple relational structure. Does this structure contain a large simple substructure? The answer for binary relational structures[1] is provided by the following theorem of Schmerl and Trotter.

Theorem 3.2.1 (Schmerl and Trotter [129])**.** *Every simple binary relational structure on $n > 2$ elements contains a simple binary relational structure on $n-1$ or $n-2$ elements.*

A version of this theorem for related objects known as 2-structures was given by Ehrenfeucht and Rozenberg [59] (see also the book by Ehrenfeucht, Harju, and Rozenberg [56]), while a proof for the special case of permutations can be found in Murphy's thesis [112].

Surprisingly, given the vast amount of research on substitution decompositions, I do not know of a Schmerl-Trotter theorem for general relational structures. Some work has been done on a single k-ary relation though, see Ehrenfeucht and McConnell [58] and Bonizzoni and McConnell [36].

Theorem 3.2.1 gives a semi-algorithm for determining if a class (of binary relational structures) contains only finitely many simple objects. If the class $\mathcal{C}$ contains only finitely many simple structures, then clearly there is an integer n so that $\mathcal{C}$ does not contain any simple structures on $n-1$ or $n-2$ elements. In the other direction, Theorem 3.2.1 shows that if we have found such an integer n then $\mathcal{C}$ contains no simple structures on $n-2$ or more elements. Therefore, when a class happens to contain only finitely many simple structures, this fact can be verified by routine computation. The other direction remains open.

Question 3.2.2. *Is it decidable whether a finitely based class (of permutations, or graphs, or relational structures, etc.) contains infinitely many simple objects?*

An answer to Question 3.2.3 would be very useful because much can be said about

[1] A relational structure is said to be binary if every relation in its language is binary, so Theorem 3.2.1 applies to graphs, tournaments, posets, and permutations.

classes with only finitely many simple objects; we discuss the permutation case in the next subsection.

3.2.2 Classes with only finitely many simple permutations

A class with only finitely many simple permutations has a recursive structure in which long permutations are built up from smaller permutations (their intervals). Thus it is natural to expect these classes to have algebraic generating functions, and this intuition is borne out by the following theorem.

Theorem 3.2.3 (Albert and Atkinson [3]). *A permutation class with only finitely many simple permutations has an algebraic generating function.*

This theorem is made all the more impressive by the following two facts:

- By Theorem 3.2.1 if the class $\mathcal{C}$ happens to have only finitely many simple permutations, then this fact can be verified and a complete list of these simple permutations can be produced mechanically as discussed in the previous subsection.
- The proof of Theorem 3.2.3 is constructive in the sense that if one is given the basis for $\mathcal{C}$ and the list of all (finitely many) simple permutations in $\mathcal{C}$, then the algebraic generating function for $\mathcal{C}$ can be computed completely mechanically.

The canonical example of a class with only finitely many simple permutations is Av(231). By considering the entries to the left and to the right of the n in a permutation in $\mathrm{Av}_n(231)$ as we did in Subsection 1.4.1, one simultaneously derives a decomposition of these permutations that leads immediately to the Catalan numbers and sees that this class contains no simple permutations of length three[2] or longer.

Another example of a class with only finitely many simple permutations is the class of separable permutations from Subsection 1.5.4.

One of the notable features of Theorem 3.2.3 is that it does not seem to require the class to be finitely based. However, this is merely an illusion (as it must be, by Proposition 2.5.1):

[2]Actually, there are no simple permutations of length three, 231-avoiding or otherwise.

Theorem 3.2.4 (Albert and Atkinson [3], Murphy [112]). *A permutation class with only finitely many simple permutations is both finitely based and partially well-ordered.*

Theorem 3.2.3's guarantee of algebraicity can sometimes be strengthened to rationality. We have already seen one example of this in Theorem 1.4.2. We conclude with another example.

Theorem 3.2.5 (Albert and Atkinson [3]). *Every subclass of* $\mathrm{Av}(k(k-1)\cdots 21)$ *with only finitely many simple permutations has a rational generating function.*

3.2.3 The number of simple permutations

The number of simple simple permutations is given, asymptotically, by the following theorem.

Theorem 3.2.6 (Albert, Atkinson, and Klazar [5]). *The number of simple permutations of length n is*

$$\frac{n!}{e^2}\left(1-\frac{4}{n}+\frac{2}{n(n-1)}+O\left(\frac{1}{n^3}\right)\right),$$

and forms a nonholomonic sequence in n.

Here we present a probabilistic proof that gives the first term of this asymptotic expansion. We begin by observing that intervals of length longer than two are quite rare. This result is implicit in the work of Uno and Yagiura.

Proposition 3.2.7 (Uno and Yagiura [152]). *Almost no permutations of length n contain an interval of length r for any $3 \le r \le n-1$.*

Proof. Let π be a permutation of length n chosen uniformly at random, let $X^{(r)}$ denote the number of intervals of length r in π, and let $X = X^{(3)} + X^{(4)} + \cdots + X^{(n-1)}$ denote the number of intervals of lengths 3 through $n-1$ in π.

First fix two intervals $I = \{i, i+1, \ldots, i+r-1\}$ and $J = \{j, j+1, \ldots, j+r-1\}$. The probability that π maps I into J is $r!/n(n-1)\cdots(n-r+1)$. Since there are $(n-r+1)^2$ different choices for I and J, we get that

$$\mathbb{E}[X^{(r)}] = \frac{r!(n-r+1)^2}{n(n-1)\cdots(n-r+1)}. \tag{3.1}$$

We need four instantiations of (3.1):

$$\begin{aligned}
\mathbb{E}[X^{(3)}] &= \frac{6(n-2)}{n(n-1)} \leq \frac{6}{n}, \\
\mathbb{E}[X^{(4)}] &= \frac{24(n-3)}{n(n-1)(n-2)} \leq \frac{24}{n^2}, \\
\mathbb{E}[X^{(n-2)}] &= \frac{18}{n(n-1)} \leq \frac{24}{n^2} \text{ (for } n \geq 4), \\
\mathbb{E}[X^{(n-1)}] &= \frac{4}{n}.
\end{aligned}$$

We also need to estimate $\mathbb{E}[X^{(r)}]$ for $5 \leq r \leq n-3$. We compute that

$$\frac{\mathbb{E}[X^{(r+1)}]}{\mathbb{E}[X^{(r)}]} = \frac{(r+1)(n-r)}{(n-r+1)^2} \tag{3.2}$$

By setting (3.2) equal to 1 and solving using the quadratic formula, it can be seen that $\mathbb{E}[X^{(r+1)}]/\mathbb{E}[X^{(r)}]$ is unimodal for $4 \leq r \leq n-2$ when $n \geq 4$. Thus $\mathbb{E}[X^{(r)}] \leq 24/n^2$ for these values of n and r.

Now by linearity of expectation, we have that

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{r=3}^{n-1} \mathbb{E}[X^{(r)}] \\
&\leq \mathbb{E}[X^{(3)}] + (n-5)\mathbb{E}[X^{(4)}] + \mathbb{E}[X^{(n-1)}] \\
&\leq \frac{6}{n^2} + (n-5)\frac{24}{n^2} + \frac{4}{n} \\
&\to 0.
\end{aligned}$$

Finally observe that

$$\Pr[\pi \text{ has an interval of length } r \text{ for some } 3 \leq r \leq n-1] \leq \mathbb{E}[X] \to 0,$$

as desired. □

In many contexts, arguments similar to Proposition 3.2.7 can be used to show that almost every object is simple. Erdős, Fried, Hajnal, and Milner [62] give such an argument for tournaments, while Möhring [109] takes a broader look.

Returning to the permutation context, Proposition 3.2.7 shows that almost all permutations without intervals of length two are simple. Hence we may ignore longer intervals in estimates of the number of simple permutations. The proof of that proposition also shows that $\mathbb{E}[X^{(2)}]$, the expected number of intervals of length two in a permutation of length n, tends to 2.

As we are now freed from considering longer intervals, let us drop the superscript from $X^{(2)}$ and partition this random variable as $X = X_1 + X_2 + \cdots + X_{n-1}$ where X_i is the indicator variable for the event $B_i = \{\pi(i+1) = \pi(i) \pm 1\}$. In other words, B_i is the event that $\pi(i)$ and $\pi(i+1)$ form an interval of length two.

Since the X_i's are mostly independent and their sum has finite expectation, one might expect X to tend to the Poisson distribution. This would imply that $X = 0$ with probability tending to e^{-2}. Indeed this is the case, as was shown by Wolfowitz and Kaplansky in the 1940's[3]. Recently, Corteel, Louchard, and Pemantle gave a more modern proof. Here we give yet another proof, using Brun's Sieve. In the statement of Brun's Sieve we keep our set-up from the previous paragraph (that X is the sum of the indicator variables for $B_1, B_2, \ldots, B_{n-1}$), and introduce the quantity $S^{(k)} = \sum \Pr[B_{i_1} \wedge B_{i_2} \wedge \cdots \wedge B_{i_k}]$ where the sum is over all subsets $\{i_1, i_2, \ldots, i_k\} \subseteq [n-1]$.

Theorem 3.2.8 (Brun's Sieve; see Alon and Spencer [9, Theorem 8.3.1])**.** *Suppose that* $\mathbb{E}[X]$ *tends to a constant* μ *and that* $S^{(k)} \to \mu^k/k!$ *for every fixed* k*. Then for every natural number* t*,*

$$\Pr[X = t] \to \frac{\mu^t}{t!} e^{-\mu}.$$

In particular, if the hypotheses of Brun's Sieve hold then $X = 0$ with probability tending to $e^{-\mu}$.

Theorem 3.2.9 (Wolfowitz [162]; Kaplansky [85]; Corteel, Louchard, and Pemantle [50])**.** *Let* X *denote the number of intervals of length* 2 *in a permutation of length* n*, selected uniformly at random. Then the distribution of* X *tends to the Poisson distribution, so for any natural number* t*,* $\Pr[X = t] \to 2^t/e^2 t!$.

Proof. In order to use Brun's Sieve we have to establish that $S^{(k)} \to \mu^k/k!$ for every

[3]Kaplansky and Wolfowitz were studying runs, not intervals. A *run* in the permutation π is a set of consecutive indices $i, i+1, \ldots, i+r$ so that $|\pi(i+j+1) - \pi(i+j)| \leq 1$ for all $j \in [r-1]$. Thus every run is an interval, and runs of length 2 are precisely intervals of length 2.

fixed k. We first consider one summand in $S^{(k)}$, say $\Pr[B_{i_1} \wedge B_{i_2} \wedge \cdots \wedge B_{i_k}]$. Recall that B_{i_j} is the event that $\pi(i_j)$ and $\pi(i_j+1)$ form an interval of length two. Since k is fixed and n is tending to infinity, there are several liberties we may take. In particular, we may assume with probability 1 that every $\pi(i_j)$ lies in $[2, n-1]$ and that $|\pi(i_j) - \pi(i_\ell)| \geq 3$ for all $j \neq \ell \in [k]$. We may also ignore the effect of terms that do not satisfy $|i_j - i_\ell| \geq 3$ for all $j \neq \ell \in [k]$.

Now we expand $\Pr[B_{i_1} \wedge B_{i_2} \wedge \cdots \wedge B_{i_k}]$:

$$\Pr[B_{i_1} \wedge B_{i_2} \wedge \cdots \wedge B_{i_k}] = \Pr[B_{i_1}] \Pr[B_{i_2}|B_{i_1}] \cdots \Pr[B_{i_k}|B_{i_1} \wedge B_{i_2} \wedge \cdots \wedge B_{i_{k-1}}]. \quad (3.3)$$

Now consider each $\pi(i_j)$ for $j \in [k]$ as fixed. In order for B_{i_1} to hold, there are precisely two values $\pi(i_1+1)$ may have: $\pi(i_1) - 1$ or $\pi(i_2+1) = \pi(i_1) + 1$. Since $\pi(i_1+1)$ cannot equal any of the $\pi(i_j)$'s, and by our assumptions above, we get that $\Pr[B_{i_1}] = 2/(n-k)$. Continuing in this manner, $\Pr[B_{i_2}|B_{i_1}] = 2/(n-k-1)$ and in general,

$$\Pr[B_{i_j}|B_{i_1} \wedge B_{i_2} \wedge \cdots \wedge B_{i_{j-1}}] = \frac{2}{n-k-j+1},$$

so using (3.3),

$$\Pr[B_{i_1} \wedge B_{i_2} \wedge \cdots \wedge B_{i_k}] = \prod_{j=1}^{k} \frac{2}{n-k-j+1}.$$

Thus we get that

$$\sum_{\{i_1,i_2,\ldots,i_k\}\subseteq[n-1]} \Pr[B_{i_1} \wedge B_{i_2} \wedge \cdots \wedge B_{i_k}] \to \binom{n-1}{k} \prod_{j=1}^{k} \frac{2}{n-k-j+1} \to \frac{2^k}{k!},$$

as desired. □

We therefore get the first term of Theorem 3.2.6 as a corollary: the number of simple permutations is asymptotic to $n!/e^2$.

3.3 Atomicity

3.3.1 Characterization

We begin with a simple question about permutation classes: what permutation classes cannot be written as a union of two proper subclasses? Such classes are called *atomic*.

For example, Av(21) is clearly atomic since it contains only one permutation of each length, the increasing permutation. On the other hand, Av(132, 213, 231, 312) contains two permutations of each length, the increasing permutation and the decreasing permutation, and thus we have

$$\text{Av}(132, 213, 231, 312) = \text{Av}(12) \cup \text{Av}(21).$$

A natural sufficient condition for atomicity is the *joint embedding property*. A class $\mathcal{C}$ satisfies the joint embedding property if for all $\sigma, \tau \in \mathcal{C}$ there is a $\pi \in \mathcal{C}$ with $\pi \geq \sigma, \tau$. To see that this condition is sufficient for atomicity, suppose to the contrary that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ where $\mathcal{C}_1$ and C_2 are proper subclasses of $\mathcal{C}$. Now take $\sigma \in \mathcal{C}_1 \setminus \mathcal{C}_2$ and $\tau \in \mathcal{C}_2 \setminus \mathcal{C}_1$. Using joint embedding there is a permutation $\pi \in \mathcal{C}$ with $\pi \geq \sigma, \tau$. But π must lie at least one of $\mathcal{C}_1$ or $\mathcal{C}_2$, and in either case we get a contradiction.

Joint embedding is also easily shown to be necessary for atomicity. Suppose that $\mathcal{C}$ is an atomic class that fails joint embedding, so there are $\sigma, \tau \in \mathcal{C}$ so that no $\pi \in \mathcal{C}$ contains both of them. Then $\mathcal{C} = (\mathcal{C} \cap \text{Av}(\sigma)) \cup (\mathcal{C} \cap \text{Av}(\tau))$, and since both of these are proper subclasses of $\mathcal{C}$, $\mathcal{C}$ is not atomic.

This gives one characterization of atomic classes, but there is a much more interesting characterization. Since permutation classes are countable, we can write $\mathcal{C} = \{\sigma_1, \sigma_2, \sigma_3, \dots\}$. Now set π_1 equal to the empty permutation and use joint embedding to inductively select $\pi_{k+1} \in \mathcal{C}$ so that $\pi_{k+1} \geq \pi_k, \sigma_k$. Thus π_k contains $\sigma_1, \sigma_2, \dots, \sigma_{k-1}$, possibly in addition to other permutations from $\mathcal{C}$. If we now take the union of the π_n's in an appropriate manner, then we will have constructed in some sense an "infinite permutation" containing every element of our atomic class (and nothing else). This union is constructed explicitly by Atkinson, Murphy, and Ruškuc [15] in the permutation case. Instead, we will return to the context of relational structures, where many of the technicalities disappear.

We define atomicity and joint embedding analogously in the general case. Let $\mathcal{L}$ be a relational language and let $\mathcal{C}$ be a closed class of $\mathcal{L}$-structures. Then $\mathcal{C}$ is said to be atomic if it cannot be written as a union of two proper subclasses, and it is said to have the joint embedding property if for all $\mathcal{A}, \mathcal{A}' \in \mathcal{C}$ there is a $\mathcal{B} \in \mathcal{C}$ with $\mathcal{A}, \mathcal{A}' \leq \mathcal{B}$. For a (possibly infinite) $\mathcal{L}$-structure $\mathcal{B}$, the *age of* $\mathcal{B}$ is the set

$$\text{Age}(\mathcal{B}) = \{\text{isomorphism types of finite } \mathcal{L}\text{-structures } \mathcal{A} \leq \mathcal{B}\},$$

and say that a class of relational structures is an age if it is the age of some relational structure.

We have already seen numerous examples of ages; in Subsection 1.6.1 we introduced the age of a vector in $(\mathbb{N} \cup \infty)^m$, while in Subsection 1.5.5 we studied permutations that embed into the increasing oscillating sequence, which is (although we did not make this point at the time) an age. We will discuss a few more examples in Subsection 3.3.3, but first we characterize ages.

The following theorem is due to Fraïssé [68]. Rediscoveries (in various contents) can be found in Atkinson, Murphy, and Ruškuc [15]; Schneinerman [128]; and Usha Devi and Vijayakumar [153].

Theorem 3.3.1 (Fraïssé [68]; see also Hodges [81, Section 7.1])**.** *Let $\mathcal{L}$ be a finite relational language and $\mathcal{C}$ a countable closed class of $\mathcal{L}$-structures. The following are equivalent:*

1. *$\mathcal{C}$ is atomic,*

2. *$\mathcal{C}$ satisfies the joint embedding property,*

3. *$\mathcal{C} = \text{Age}(\mathcal{B})$ for some countable $\mathcal{L}$-structure $\mathcal{B}$.*

Proof. Our proof of (1) $\iff$ (2) for permutation classes generalizes with absolutely no difficulty, so we have only to show (2) $\iff$ (3), of which one direction is trivial: (3) $\implies$ (2).

To prove (2) $\implies$ (3), suppose that $\mathcal{C}$ satisfies the joint embedding property, and list the elements of $\mathcal{C}$ as $\mathcal{A}_1, \mathcal{A}_2, \ldots$ (that $\mathcal{C}$ is countable follows from the fact that it has a finite language). Set $\mathcal{B}_1 = \mathcal{A}_1$. For each i use joint embedding to find a structure $\mathcal{B}'_{i+1} \in \mathcal{C}$ containing both $\mathcal{B}_i$ and $\mathcal{A}_{i+1}$. Take $\mathcal{B}_{i+1}$ to be isomorphic to $\mathcal{B}'_{i+1}$ so that the embedding $\varphi : \text{dom}(\mathcal{B}_i) \to \text{dom}(\mathcal{B}_{i+1})$ is the identity (to accomplish this we only have to relabel the domain of B'_{i+1}). Finally set $\mathcal{B} = \bigcup_{i \in \mathbb{N}} \mathcal{B}_i$. ☐

A much stronger condition than atomicity is that of *homogeneity*. Such classes satisfy not only the joint embedding property, but also the *amalgamation property*; the homogeneous permutation classes are characterized in Cameron [41].

We close with a question that appears to be quite difficult.

Question 3.3.2. *Is it decidable (from the basis) whether a finitely based class of relational structures is atomic?*

3.3.2 Non-atomic classes

The best one might hope to say about non-atomic classes in general is that they can be expressed as a union of a finite number of atomic classes[4]. However, Theorem 3.3.1 and the fact that there are infinite antichains of permutations quickly eliminate this possibility: Let A denote an infinite antichain of permutations and consider the class $\mathcal{C}$ containing every permutation contained in an element of A. This class cannot be expressed as a finite union of atomic class because then one of the classes would have to contain more than one element of A, and thus would not satisfy joint embedding.

In the other direction, Proposition 1.6.4 extends to this level of generality:

Proposition 3.3.3. *Every pwo closed class of relational structures can be expressed as a finite union of atomic classes.*

3.3.3 Examples of atomic permutation classes

In Subsection 1.6.1 we introduced the age of a vector in $(\mathbb{N} \cup \infty)^m$, and it is easy to see that these are the only atomic classes of vectors. For permutations, the issue is more interesting.

Suppose that the permutation class $\mathcal{C}$ is atomic. Then by Theorem 3.3.1, the relational structures corresponding to permutations in $\mathcal{C}$ are precisely the relational structures lying in Age($\mathcal{B}$) for a countable $\{<, \prec\}$-structure $\mathcal{B}$ where both $<^{\mathcal{B}}$ and $\prec^{\mathcal{B}}$ are linear orders on dom($\mathcal{B}$). For this reason, one might say that $\mathcal{B}$ consists of two linear

[4]Indeed, Rao [122] conjectured precisely this for the context of graphs, which was the inspiration for Usha Devi and Vijayakumar's rediscovery of Theorem 3.3.1.

orders — $(\mathrm{dom}(\mathcal{B}), <^{\mathcal{B}})$ and $(\mathrm{dom}(\mathcal{B}), \prec^{\mathcal{B}})$ — together with an identification of their domains. Working from the other direction, let $(\leq^X, X)$ and $(\leq^Y, Y)$ be any two linear orders and suppose that we have a bijection $f : X \to Y$. We can then form a $\{<, \prec\}$-structure $\mathcal{B}$ over X where $x \leq^{\mathcal{B}} y \iff x \leq^X y$ and $x \prec^{\mathcal{B}} y \iff f(x) <^Y f(y)$. Therefore, in the permutation context, we can view atomic classes as being *ages of bijections between two linear orders*. When we wish to emphasize the domain and range of this bijection, we will use the notation $\mathrm{Age}(f : (X, \leq^X) \to (Y, \leq^Y))$.

For example, in Subsection 1.5.5, we considered the class of permutations that embed into the increasing oscillating sequence. As we defined it, the increasing oscillating sequence can be viewed as a bijection $f : \mathbb{N} \to \mathbb{N} \setminus \{2\}$ where $f(i)$ is the ith element of the sequence. The age of f is then the class of permutations that embed into the increasing oscillating sequence.

Here we present a short list of examples of other atomic class of permutations:

1. The set of all permutations. Here one can either use the joint embedding property or construct an atomic representation. In fact, the age of any bijection from $\mathbb{Q}$ to $\mathbb{N}$ is the set of all permutations (this is a special case of the upcoming Theorem 3.3.5).

2. All $\oplus$-complete and $\ominus$-complete classes, because they have the joint embedding property.

3. In particular, all singleton-based classes, because they are either $\oplus$-complete or $\ominus$-complete (see Section 1.5).

4. The skew-merged permutations from Subsection 1.7.1, also by joint embedding.

5. The W-classes from Section 1.7.2. Two examples are shown in Figure 3.3.

3.3.4 Restricting the domain and range

Whereas we have mostly been discussing concepts that extend to many other contexts, here we briefly discuss a question which can really only be asked for permutations.

Define $\mathcal{T}(X, Y)$ as the set of all (atomic) classes of permutations that can be expressed as $\mathrm{Age}(f : X \to Y)$. There are now several natural questions:

- For a given X and Y, what is $\mathcal{T}(X, Y)$?

- What conditions on X, Y, W, and Z would allow $\mathcal{T}(X, Y) \subseteq \mathcal{T}(W, Z)$?

- Can it be decided from its basis if a given class lies in $\mathcal{T}(X, Y)$?

For the first two questions there are some partial answers. The characterization of $\mathcal{T}(\mathbb{N}, \mathbb{N})$, the *natural classes*, is almost completely provided by the following theorem.

Theorem 3.3.4 (Atkinson, Murphy, and Ruškuc [15]). *Let $\mathcal{C} \in \mathcal{T}(\mathbb{N}, \mathbb{N})$ be a finitely based permutation class. Then either*

1. *$\mathcal{C} = \mathrm{Age}(\pi) \oplus \mathcal{D}$ for a finite permutation π and a $\oplus$-complete permutation class $\mathcal{D}$, both determined uniquely by $\mathcal{C}$, or*

2. *$\mathcal{C} = \mathrm{Age}(f : \mathbb{N} \to \mathbb{N})$ for an ultimately periodic f, meaning that there are $N, m > 0$ such that $f(n + m) = f(n) + m$ for all $n \geq N$.*

Ruškuc and Huczynska investigated the *supernatural classes*, those that lie in $\mathcal{T}(X, \mathbb{N})$. Before mentioning their results, we need a bit more notation. Given two linear orders $(X, \leq^X)$ and $(Y, \leq^Y)$ we define $X \oplus Y$, the *direct sum* of X and Y, to be the disjoint union $X \uplus Y$ with the ordering $x \leq y$ if $x, y \in X$ and $x \leq^X y$ or if $x, y \in Y$ and $x \leq^Y y$ or if $x \in X$ and $y \in Y$. We also let $-\mathbb{N}$ denote the nonpositive integers with their standard order, and for a linear order X and integer k we denote by kX the direct sum of k copies of X.

With this notation established, we divide linear orders into two types.

- Type 1: a linear order is of type 1 if it contains either $k\mathbb{N}$ or $k(-\mathbb{N})$ for all natural numbers k.

- Type 2: a linear order is of type 2 if it contains $k\mathbb{N}$ and $k(-\mathbb{N})$ for only finitely many values of k.

We begin with the characterization problem for type 1 orders.

Theorem 3.3.5 (Huczynska and Ruškuc [82]). *Let X be a type 1 linear order. Then $\mathcal{T}(X, \mathbb{N})$ contains precisely one class, the class of all permutations.*

The situation for type 2 orders is quite different.

Theorem 3.3.6 (Huczynska and Ruškuc [82]). *Let X and Y be two nonisomorphic type 2 linear orders. Neither one of $\mathcal{T}(X,\mathbb{N})$ or $\mathcal{T}(Y,\mathbb{N})$ is contained in the other.*

We conclude with one of the more intriguing open problems about supernatural classes. Huczynska and Ruškuc proved the following interpolation result: if an atomic class lies in both $\mathcal{T}(\mathbb{N},\mathbb{N})$ and $\mathcal{T}(\ell\mathbb{N},\mathbb{N})$, then that class also lies in $\mathcal{T}(k\mathbb{N},\mathbb{N})$ for all $k \in [\ell]$. One would naturally expect this behavior to extend to the following generalization, but it remains open.

Question 3.3.7 (Huczynska and Ruškuc [82]). *If $\mathcal{C} \in \mathcal{T}(j\mathbb{N},\mathbb{N}) \cap \mathcal{T}(\ell\mathbb{N},\mathbb{N})$ for some $j \leq \ell$, does $\mathcal{C}$ also lie in $\mathcal{T}(k\mathbb{N},\mathbb{N})$ for all $j \leq k \leq \ell$?*

3.4 Ages of finite objects

In Section 3.3 we concentrated on ages of infinite objects. While these are the most interesting ages from a structural point of view, there is a tantalizing question to be raised about ages of a finite objects. Recall that the sequence $a_0, a_1, \ldots, a_n$ is said to be *unimodal* if it increases and then decreases, or in other words, if there is an index k for which $a_0 \leq a_1 \leq \cdots \leq a_k$ and $a_k \geq a_{k+1} \geq \cdots \geq a_n$. A stronger condition (for sequences of positive numbers) is that of log-concavity; the sequence above is *log-concave* if $a_{i-1}a_{i+1} \leq a_i^2$ for all $1 \leq i \leq n-1$. Similarly, a polynomial is said to be unimodal (resp. log-concave) if its sequence of coefficients is unimodal (resp. log-concave).

We begin with permutations. Figure 3.4 shows the Hasse diagram for Age(31524). The generating function for this age is $1 + x + 2x^2 + 5x^3 + 5x^4 + x^5$. Let us say that a poset is rank-unimodal (resp. rank-log-concave) if its generating function is unimodal (resp. log-concave), so our example is rank-unimodal. However, Age(π) is not rank-log-concave for almost all permutations because their generating functions begin $1+x+2x^2+\cdots$ (and even ignoring this trivial violation, there are non-log-concave ages of permutations, for example, our previous example of Age(31524), because $1 \cdot 5 > 2^2$). A check of all permutations of length at most 9 has failed to produce a counterexample to the following question.

Question 3.4.1. *Is* Age(π) *rank-unimodal for all permutations* π*?*

Obviously we can adapt this question to the other contexts, and there are both positive and negative results (and open problems). We begin with words.

Theorem 3.4.2 (Chase [42])**.** *For all words* w*,* Age(w) *is not only rank-unimodal, but rank-log-concave.*

It is worth observing that Chase's proof of this theorem uses a fair amount of algebraic manipulation. A combinatorial proof would instead construct an injection from pairs of subwords of w of lengths $i-1$ and $i+1$ to pairs of subwords of w both of length i. Despite Stanley's request for such a proof in [143], one has yet to be found.

The composition version of this question also seems to be open. Via the correspondence between compositions and layered permutations discussed in Subsection 1.6.3, this is a special case of Question 3.4.1.

Question 3.4.3. *For a composition* w*, is* Age(w) *rank-unimodal?*

These very similar questions have a natural generalization to the age of any finite relational structure, and this general version of the question was posed by Pouzet and Rosenberg [119].

Unfortunately, such a generalization would have to include partitions. (In fact, even a weaker conjecture only about relations, not even relational structures, would include partitions.) Stanton [147] observed that the age of the partition $(8, 8, 4, 4)$ is *not* rank-unimodal. (On the other hand, the age of the *composition* $(8, 8, 4, 4)$ *is* rank-unimodal.) Also, alarmingly, this is the smallest partition with a non-unimodal age, and it is a partition of 24, a fact that certainly weakens the computational evidence for Question 3.4.1.

On the other hand, let (n^m) denote the partition with m entries, each equal to n. The age of this partition, or in other words, the interval $[1, (n^m)]$ in Young's lattice, is written $L(m, n)$. Figure 3.5 shows the Hasse diagram for $L(2, 3)$. Sylvester showed that these ages are unimodal.

Theorem 3.4.4 (Sylvester [148])**.** *For all* m *and* n*,* $L(m, n)$ *is rank-unimodal.*

Since Sylvester's proof, numerous other proofs have been given. Stanley [141] proved Theorem 3.4.4 using the hard Lefschetz theorem of algebraic geometry. In fact, Stanley's proof yielded much more; he was the first to establish that these lattices have what is known as the *strong Sperner property*. Later Proctor [121] stripped away much of the complexity of Stanley's argument, leaving a proof that requires only linear algebra. It was not until 1990 that O'Hara [117] discovered a long sought-after combinatorial proof of Theorem 3.4.4.

Although the overly ambitious conjecture of Pouzet and Rosenberg fell at the hands of Stanton's examples, they did prove an important special case.

Theorem 3.4.5 (Pouzet and Rosenberg [119]). *Let $\mathcal{A}$ be a finite relational structure. The first half (rounded up) of the coefficients of the generating function for* $\mathrm{Age}(\mathcal{A})$ *are increasing.*

This theorem is quite powerful. For example, it is easy to see that $L(m, n)$ is rank-symmetric (meaning in this case that there are just as many partitions of i in $L(m, n)$ as there are partitions of $mn - i$), so Theorem 3.4.5 gives yet another proof of Theorem 3.4.4. Stanley takes up their approach and simplifies it for the case of $L(m, n)$ in [142]; interestingly, he obtains via his machinery another proof of a result of Lovász [97] about the edge-reconstruction conjecture for graphs.

We conclude with another result of Pouzet and Rosenberg, even though it runs counter to the title of this section.

Theorem 3.4.6 (Pouzet and Rosenberg [119]). *For any relational structure $\mathcal{A}$ over an infinite ground set, the coefficients of the generating function for* $\mathrm{Age}(\mathcal{A})$ *are increasing.*

Chapter 4

Generating trees

Notice: The work in this chapter is adapted from Vatter [155].

4.1 Basic definitions

The oldest and most established technique for the systematic enumeration of permutation classes is that of *generating trees*, introduced by Chung, Graham, Hoggatt, and Kleiman [48] and used extensively by many others[1] since. The recently introduced ECO (enumerating combinatorial objects) method[2] extends the notion of generating trees to other combinatorial contexts. There has also been some interest in the algebraic properties of generating trees and ECO systems[3].

A generating tree is a rooted, labeled, and typically infinite tree such that the label of a node determines the labels of its children. Sometimes the labels of the tree are taken to be natural numbers, but this is not necessary, and the algorithm we will describe labels nodes by permutations. Therefore we specify a generating tree by supplying the label of the root and a set of *succession rules*. For example, the complete binary tree may be given by

$$\begin{array}{ll} \text{Root:} & (SS) \\ \text{Rule:} & (SS) \rightsquigarrow (SS)(SS). \end{array}$$

[1]To name a few: Barcucci, Del Lungo, Pergola, and Pinzani [22, 23]; Chen, Mansour, and Yan [43]; Chow and West [44]; Dulucq, Gire, and Guibert [54]; Dulucq, Gire, and West [55]; Guibert and Pergola [77]; Kremer [91]; Kremer and Shiu [92]; Marinov and Radoičić [106]; Merlini, Sprugnoli, and Verri [108]; Pergola and Sulanke [118]; Stankova [135, 138]; Stankova and West [137]; and West [158, 159].

[2]See Barcucci, Del Lungo, Pergola, and Pinzani [21] for a survey, and Barcucci, Pergola, Pinzani, and Rinaldi [24] or Ferrari, Pinzani, and Rinaldi [64] for other applications.

[3]For this the reader is referred to Duchi, Fedou, and Rinaldi [53]; Ferrari, Pergola, Pinzani, and Rinaldi [63]; and Merlini, Sprugnoli, and Verri [107].

The connection to permutation classes comes through *pattern-avoidance trees*. We say that the permutation σ of length n is a *child* of $\pi \in S_{n-1}$ if σ can be obtained by inserting n into π. This defines a rooted tree T on the set of all permutations. For a basis B, we define the pattern-avoidance tree $T(B)$ to be the subtree of T whose nodes are the permutations in the class $\mathrm{Av}(B)$. For example, the first four levels of $T(132, 3241)$ are shown in Figure 4.1.

The *active sites* (relative to B) of $\pi \in \mathrm{Av}_{n-1}(B)$ are the positions i for which inserting n right before the ith entry of π produces a permutation in $\mathrm{Av}(B)$. By convention, $n+1$ is an active site of π if appending n to the end of π produces a permutation in $\mathrm{Av}(B)$. An inactive site is any site that is not active. For example, the active sites of 213 relative to $B = \{132, 3241\}$ are 1 and 4, whereas the inactive sites are 2 and 3.

In order to enumerate the permutation class $\mathrm{Av}(B)$, we want to find a generating tree isomorphic (as a rooted tree) to $T(B)$. Let $T(B;\pi)$ denote the subtree of $T(B)$ that is rooted at π and contains all descendants of π. In an isomorphism between $T(B)$ and a generating tree, every permutation of $\mathrm{Av}(B)$ is assigned a label. Clearly two permutations π and σ may be assigned the same label if and only if $T(B;\pi)$ and $T(B;\sigma)$ are isomorphic (again, as rooted trees). Thus each pattern-avoidance tree $T(B)$ is isomorphic to a canonical generating tree whose labels correspond exactly to the isomorphism classes of $\{T(B;\pi) : \pi \in \mathrm{Av}(B)\}$.

In particular, $T(B)$ is isomorphic to a finitely labeled generating tree if and only if the set of all principal subtrees $\{T(B;\pi) : \pi \in \mathrm{Av}(B)\}$ contains only finitely many isomorphism classes. When this occurs, $\mathrm{Av}(B)$ has a rational generating function which may be routinely computed using the transfer matrix method (see Stanley [144, Section 4.7] for details, or the next section for an example). We characterize these classes in Theorem 4.6.2.

4.2 Examples

We begin with one of the simplest examples of the isomorphism between a pattern-avoidance tree and a generating tree.

Proposition 4.2.1. *The pattern-avoidance tree* $T(132, 231)$ *is isomorphic to the complete binary tree.*

Proof. Take $\pi \in \mathrm{Av}_{n-1}(132, 231)$. We may obtain a permutation in $\mathrm{Av}_n(132, 231)$ by inserting n either at the beginning or the end of π, but nowhere in between, so every node of $T(132, 231)$ has precisely two children, as desired. $\square$

We move on to a slightly more complicated example where more than one label is needed.

Proposition 4.2.2. *The pattern-avoidance tree* $T(132, 3241)$ *is isomorphic to the generating tree given by*

$$\begin{array}{llll} \textit{Root:} & (2) & & \\ \textit{Rules:} & (2) & \rightsquigarrow & (2)(3) \\ & (3) & \rightsquigarrow & (2)(3)(3). \end{array}$$

Proof. Take a permutation $\pi \in \mathrm{Av}_{n-1}(132, 3241)$. There are at most three sites in which we may insert n to form a $\{132, 3241\}$-avoiding child: the beginning, the end, and the site directly to the right of $n-1$. Indeed, if we insert n to the left of $n-1$ but not at the beginning we form a 132-pattern, while if we insert n further than one site to the right of $n-1$ but not at the end we get a subsequence $n-1, x, n, y$ and either $x < y$, giving a 132-pattern, or $x > y$, giving a 3241-pattern. We can therefore constuct an isomorphic generating tree with just two labels, label (2) for the nodes where $n-1$ is the last entry and label (3) for the nodes where $n-1$ is not the last entry, proving the proposition. $\square$

Sometimes, of course, we cannot make do with only finitely many labels, as witnessed by the following example.

Proposition 4.2.3. *The pattern-avoidance tree* $T(123)$ *is isomorphic to the generating tree given by*

$$\begin{array}{lll} \textit{Root:} & (2) & \\ \textit{Rules:} & (j) & \rightsquigarrow \quad (2)(3)\cdots(j+1). \end{array}$$

Proof. Take $\pi \in \mathrm{Av}_{n-1}(123)$ and let $j_\pi = \min\{i : \pi(i) < \pi(i+1)\}$ (so j is the position of the first ascent in π), or set $j_\pi = n$ if π does not contain an ascent (so $\pi = (n-1)\cdots 21$). The active sites of π are then precisely $\{1, 2, \ldots, j_\pi\}$. Now let $\sigma \in \mathrm{Av}_n(123)$ denote a child of π. If σ was formed by inserting n right before the ith element of π for some $2 \le i \le j_\pi$ then $j_\sigma = i$. On the other hand, if σ was formed by inserting n at the beginning of π then $j_\sigma = j_\pi + 1$. This proves the isomorphism, because we can map π to a node labeled (j_π). ◻

A more complicated example is $T(1234)$, first described by West [158]. This tree is isomorphic to generating tree defined by

$$\begin{array}{lll} \text{Root:} & (2,2) & \\ \text{Rule:} & (s,t) & \rightsquigarrow \quad (2,t+1)(3,t+1)\cdots(s,t+1)(s,s+1)(s,s+2)\cdots(s,t)(s+1,t+1). \end{array}$$

While verifying this isomorphism is not difficult (consider the lexicographically first ascent and the lexicographically first occurrence of 123 in the permutation), it is much harder to obtain the generating function for Av(1234) from this tree; for the details of this see Bousquet-Mélou [38].

Upon finding a generating tree isomorphic to $T(B)$, one often wishes to get the generating function for $\mathrm{Av}(B)$. In general, as witnessed by Bousquet-Mélou [38] and Banderier, Bousquet-Mélou, Denise, Flajolet, Gardy, and Gouyou-Beauchamps [20], this process can be quite intricate. Therefore we will give an example with only finitely many labels, where the process is routine.

Proposition 4.2.4. *The generating function for* $\mathrm{Av}(132, 3241)$ *is*

$$\frac{1-2x}{1-3x+x^2}.$$

Proof. In Proposition 4.2.2 we showed that $T(132, 3241)$ is isomorphic to the generating tree

$$\begin{array}{llcl} \text{Root:} & (2) & & \\ \text{Rules:} & (2) & \rightsquigarrow & (2)(3) \\ & (3) & \rightsquigarrow & (2)(3)(3). \end{array}$$

Let $f_{(2)}(x)$ denote the height-generating function for the nodes labeled by (2) and $f_{(3)}(x)$ be the height-generating function for the nodes labeled by (3), so, to be clear,

$$f_{(2)}(x) = \sum_{n=0}^{\infty} (\#\text{ nodes labeled by (2) on level } n)\, x^n.$$

Then we have that

$$\begin{aligned} f_{(2)} &= xf_{(2)} + xf_{(3)} + x, \\ f_{(3)} &= xf_{(2)} + 2xf_{(3)}. \end{aligned}$$

(The x term above counts the permutation 1.) Translating this into linear algebra,

$$\begin{pmatrix} f_{(2)} \\ f_{(3)} \end{pmatrix} = \begin{pmatrix} x & x \\ x & 2x \end{pmatrix} \begin{pmatrix} f_{(2)} \\ f_{(3)} \end{pmatrix} + \begin{pmatrix} x \\ 0 \end{pmatrix},$$

and solving this gives

$$\begin{pmatrix} f_{(2)} \\ f_{(3)} \end{pmatrix} = \begin{pmatrix} \frac{x-2x^2}{1-3x+x^2} \\ \frac{x^2}{1-3x+x^2} \end{pmatrix}.$$

The generating function for $\mathrm{Av}(132, 3241)$ is then $f_{(2)} + f_{(3)} + 1$ (the 1 counts the empty permutation). □

4.3 Generating trees with bounded degrees

We now begin working towards our characterization of the finitely labeled generating trees. One requirement for $T(B)$ to be isomorphic to a finitely labeled generating tree is that there must be a bound on the number of children a node of $T(B)$ may have. For this to occur, B must contain both a child of an increasing permutation (such as 132, 4123, or 12345) and a child of a decreasing permutation (such as 231, 3241, or 54321), because otherwise either $12 \cdots n$ or $n \cdots 21$ will have $n + 1$ children for all n. In fact, Kremer and Shiu [92] showed that this is enough. We include a short proof below.

Theorem 4.3.1 (Kremer and Shiu [92]). *The pattern-avoidance tree $T(B)$ has bounded degrees if and only if B contains both a child of an increasing permutation and a child of a decreasing permutation.*

Proof. We have already noted that the condition on B is necessary, so it suffices to show that it is sufficient. Assume not, so that although B contains a child of $12\cdots k$ and a child of $\ell\cdots 21$, $T(B)$ does not have bounded degrees. Set $n = (k-1)(\ell-1)+1$.

We claim that there is a permutation $\pi \in T(B)$ of length n with $n+1$ children (or, equivalently, $n+1$ active sites). Since $T(B)$ does not have bounded degrees, we can find a permutation in $T(B)$ with at least $n+1$ active sites. Suppose that $i_1 < i_2 < \cdots < i_{n+1}$ are active sites of π. Now form the word $w = \pi(i_1)\pi(i_2)\cdots\pi(i_n)$ and set $\sigma = \mathrm{st}(w)$. For example, suppose that $n = 3$ and $\pi = 461523$ with active sites $\{1, 2, 5, 7\}$. Then we get $w = \pi(1)\pi(2)\pi(5) = 462$ and $\sigma = \mathrm{st}(462) = 231$.

By construction, σ is a permutation of length n with $n+1$ children in $T(B)$, proving the claim. However, by our choice of n, the Erdős-Szekeres theorem shows that σ contains either an increasing subsequence of length at least k or a decreasing subsequence of length at least ℓ. Thus we have reached a contradiction because our assumptions on B imply that at least one of the children of σ must contain a pattern from B and thus cannot be a node of $T(B)$. □

In general, trees with bounded degrees need not be isomorphic to finitely labeled generating trees. For example, consider the generating tree (pictured in Figure 4.2) given by

$$\begin{array}{lll} \text{Root:} & (1,1) & \\ \text{Rules:} & (i,j) & \rightsquigarrow\ (i,j-1) \text{ if } j \geq 2, \\ & (i,1) & \rightsquigarrow\ (i+1,i+1)(0), \end{array}$$

where nodes labeled by (0) do not produce children. This tree is clearly not isomorphic to a finitely labeled generating tree since the distance between two nodes of degree 2 is unbounded, but each of its nodes has at most two children. Our characterization result, Theorem 4.6.2, says that if B is finite and $T(B)$ has bounded degrees then $T(B)$ is isomorphic to a finitely labeled generating tree, so this example shows that our proof will need to make use of the special properties of pattern-avoidance trees.

In the next section we introduce lemmas and notation for pattern-avoidance trees. Section 4.5 describes our labeling algorithm, while the proof that this algorithm works is contained in Section 4.6.

4.4 Removable & GT-reducible entries

In order to motivate our technique we begin by returning to the example of $T(132, 3241)$. Since our approach in Proposition 4.2.2 was rather ad hoc, we now attempt to analyze this tree (or rather, two of its nodes) in a more systematic manner. Consider 213. In this permutation we can insert new maximal entries at the beginning or the end, but we will never be able to insert a new maximal entry between the 2 and the 3. Now let π denote a descendant of 213. From our previous comments, 213 appears as a contiguous block in π. Observe that for any possible 132 or 3142-pattern in π involving the 1 there is another pattern which uses the 2 instead. Therefore it does not matter whether or not the 1 is stuck between the 2 and the 3, and we can assign the same label to 213 as we assign to 12. In fact, a similar argument shows that 12 can be labeled by the same label as 1 receives. It is notions like these that we aim to formalize in this section.

First, we say that an entry x in the permutation π is *removable* (relative to a basis B) if it is adjacent to at most one active site. For example, every entry of 213 is removable when $B = \{132, 3241\}$, although no entry of 21 is removable. When more detail is needed, we say that the entry $\pi(i)$ is left-removable if i is an inactive site of π and right-removable if $i+1$ is inactive, so every removable entry is either left-removable, right-removable, or both.

If x is an entry of the word w and w contains distinct integers then we write $w - x$ to denote the word formed from w by removing x. For example, $461523 - 2 = 46153$. If X is a set of entries of w, we similarly write $w - X$ to denote the word formed by removing each entry in X.

When π is a node of $T(B)$ we let $T(B; \pi)$ denote the subtree consisting of π and its descendants. In order to avoid the shifting of indices and values caused by standardization we will also make use of the similar but less natural tree $W(B;n,u)$, which we define whenever u is a B-avoiding word containing distinct integers all at most n. The root of $W(B;n,u)$ is u, and it contains all B-avoiding words w that can be formed by shuffling u with a permutation on $[n+1, n+|w|-|u|]$. If v and w are nodes of $W(B;n,u)$ and w has greatest entry m, then w is a child of v if w can be obtained by inserting m into v, that is, if $v = w - m$. An example is shown in Figure 4.3. For a word $w \in W(B;n,u)$ we define active sites, inactive sites, removability, left-removability, and right-removability as we did for the permutations of $T(B)$.

Once these definitions are unraveled, it is evident that $W(B;n,u) \cong T(B;\mathrm{st}(u))$ for all allowed values of n, and if π is a permutation then $W(B;|\pi|,\pi)$ and $T(B;\pi)$ are not merely isomorphic, but they are the same tree.

If u is a word of length at least two containing the entry x we define the map

$$\partial_x : W(B;n,u) \to W(B;n,u-x)$$

by $\partial_x(w) = w - x$. Our upcoming Proposition 4.4.1 shows that ∂_x is one-to-one if x is a removable entry in u. There are several different cases in the definition of the inverse map.

First suppose that $x = u(i)$ for some $i > 2$. In this case we define the map

$$\iota^-_{u,x} : W(\emptyset;n,u-x) \to W(\emptyset;n,u)$$

by letting $\iota^-_{u,x}(w)$ denote the word obtained from w by inserting x immediately to the right of the entry $u(i-1)$. If x is left-removable in u relative to B then it is easy to see that $\iota^-_{u,x}$ maps words in $\partial_x(W(B;n,u))$ to words in $W(B;n,u)$. Furthermore, if $w \in W(B;n,u-x) \setminus \partial_x(W(B;n,u))$ then $\iota^-_{u,x}(w)$ will contain at least one pattern from B.

There are three more cases for us to define this operation. If $x = u(1)$ then we simply let $\iota^-_{u,x}(w) = xw$. Similarly, if x is the last entry of u we let $\iota^+_{u,x}(w) = wx$. Otherwise we let $\iota^+_{u,x}(w)$ denote the word formed from w by inserting x to the immediate left of $u(i+1)$.

Proposition 4.4.1. *Let u be a B-avoiding word of length at least two containing distinct integers all at most n and suppose that $x \in u$ is removable. Then $\partial_x : W(B;n,u) \to W(B;n,u-x)$ is one-to-one. More specifically, if x is left-removable then*

$$\iota^-_{u,x} \circ \partial_x : W(B;n,u) \to W(B;n,u)$$

is the identity, and if x is right-removable then

$$\iota^+_{u,x} \circ \partial_x : W(B;n,u) \to W(B;n,u)$$

is the identity.

Proof. As the various cases are quite similar, let us assume that $x = u(i)$ for some $i > 2$ and that x is left-removable. Take $w \in W(B;n,u)$. Since x is left-removable, w cannot contain an entry between $u(i-1)$ and x. Thus applying ∂_x removes x, but then $\iota^-_{u,x}$ inserts x immediately to the right of $u(i-1)$, restoring w. □

One of the implications of Proposition 4.4.1 is that $W(B;n,u)$ embeds into $W(B;n,u-x)$ by the map ∂_x whenever x is a removable entry of u. It sometimes happens that these two trees are isomorphic. If $W(B;n,u) \cong W(B;n,u-x)$, and x is removable, then we say that x is *generating-tree-reducible relative to B*, or for short, *GT-reducible*. In this language, our observation at the beginning of this section was that the entry 1 in 213 is GT-reducible for $B = \{132, 3241\}$. If the permutation π contains a GT-reducible entry, we also refer to π as being GT-reducible. We will introduce weaker notions of reducibility in the next chapter when we consider enumeration schemes.

Before ending our discussion of ι, let us note that in many cases these maps commute:

Observation 4.4.2. *Let u be a word containing distinct integers all at most n, let x and y be nonadjacent entries of u, and let $\delta, \epsilon \in \{+,-\}$. Then*

$$\iota^\delta_{u,x} \circ \iota^\epsilon_{u-x,y} = \iota^\epsilon_{u,y} \circ \iota^\delta_{u-y,x}$$

as maps from $W(\emptyset;n,u-x-y)$ to $W(\emptyset;n,u)$.

We would now like to show that it is possible to decide whether a removable entry is GT-reducible. For this we need two more definitions. Given a tree T, let $\ell_i(T)$ denote the number of nodes of T of height i, so $\ell_0(T) = 1$ unless T is the empty tree, $\ell_1(T)$ is the number of children of the root node, and $\ell_2(T)$ is the number of grandchildren of the root node. Also, if B is a finite basis, let $\|B\|_\infty$ denote the length of the longest pattern in B.

Proposition 4.4.3. *Let u be a B-avoiding word of length at least two containing distinct integers all at most n. The removable entry $x \in u$ is GT-reducible if and only if*

$$\ell_r(W(B; n, u)) = \ell_r(W(B; n, u - x))$$

for all $1 \leq r \leq \|B\|_\infty - 1$.

Proof. If $x \in u$ is GT-reducible then $W(B; n, u) \cong W(B; n, u - x)$ by definition, so $\ell_r(W(B; n, u)) = \ell_r(W(B; n, u - x))$ for all $r \in \mathbb{N}$. Suppose now that $\ell_r(W(B; n, u)) = \ell_r(W(B; n, u - x))$ for all $1 \leq r \leq \|B\|_\infty - 1$. Since x is removable, Proposition 4.4.1 shows that $W(B; n, u)$ embeds into $W(B; n, u - x)$ by the map ∂_x. Thus we would like to show that this map is onto.

Suppose not and choose $w \in W(B; n, u - x)$ that is not in the image of ∂_x. Let $\epsilon = -$ if x is left-removable in u. Otherwise x must be right-removable in u, and here we let $\epsilon = +$. Since $w \notin \partial_x(W(B; n, u))$, $\iota^\epsilon_{u,x}(w)$ contains a permutation from B. Choose a subword of $\iota^\epsilon_{u,x}(w)$ that standardizes to a member of B and label it v. Because w avoids B, x must be an entry of v.

Now consider the subword w' of $\iota^\epsilon_{u,x}(w)$ containing all entries that are either in u or in v. Because w' contains v, it contains a pattern from B. However, $w' - x$ is a subword of w, so it avoids B. We would like to find a word in $\iota^\epsilon_{u,x}(W(B; n, u - x))$ with these properties. We do this by "partially standardizing" w': replace the smallest entry of w' that is not in u by $n+1$, the next smallest entry of w' that is not in u by $n+2$, and so on. Label the resulting word w''. Notice that $w'' \in \iota^\epsilon_{u,x}(W(B; n, u - x))$, w'' contains a pattern from B, and $w'' - x$ is B-avoiding. These observations and Proposition 4.4.1 show that

$$\ell_{|w''|-|u|}(W(B; n, u)) < \ell_{|w''|-|u|}(W(B; n, u - x)).$$

Furthermore, since both w'' and u contain x, $|w''| \leq |u| + \|B\|_\infty - 1$, contradicing our hypotheses. □

4.5 The algorithm

Our work in the previous section suggests the following approach for finding a generating tree isomorphic to $T(B)$. If $1 \notin \text{Av}(B)$ then our task is quite easily accomplished, so let us assume that 1 avoids B. We start with a root node (1), a set $P = \{1\}$ of permutations that we have not checked for GT-reducible entries, and a set $\mathcal{R} = \emptyset$ of succession rules. Now we pick a permutation $\pi \in P$ of minimum length and check it for GT-reducible entries (we make the convention that the permutation 1 never has a GT-reducible entry).

First suppose that π is not GT-reducible, and say that its B-avoiding children are $\sigma_1, \sigma_2, \ldots, \sigma_t$. In this case we remove π from P, add its B-avoiding children to P, and add the succession rule

$$(\pi) \rightsquigarrow (\sigma_1)(\sigma_2)\cdots(\sigma_t)$$

to $\mathcal{R}$.

If instead π has a GT-reducible entry x then we again remove π from P, but now we search through our set of succession rules $\mathcal{R}$ and replace each instance of (π) by the label we have given to the node $\text{st}(\pi - x)$ (this label might not be $(\text{st}(\pi - x))$ because $\text{st}(\pi - x)$ may also have a GT-reducible entry). In other words, whenever a node labeled by (π) would have been produced, now a node labeled by the same label as $\text{st}(\pi - x)$ will be produced. This does not change the isomorphism type of the tree because

$$T(B;\pi) = W(B;|\pi|,\pi) \cong W(B;|\pi|,\pi - x) \cong T(B;\text{st}(\pi - x)).$$

We repeat this process until $P = \emptyset$. If we ever reach this state then we know that the generating tree we have produced is isomorphic to $T(B)$. We will prove shortly (Theorem 4.6.2) that we do reach this state when B contains both a child of an increasing permutation and a child of a decreasing permutation.

Before that, let us illustrate the process with the tree $T(132, 3241)$. We start with $P = \{1\}$ and $\mathcal{R} = \emptyset$. Then we choose 1 from P and note that it does not have a

GT-reducible entry (by our convention), so we remove 1 from P and add 12 and 21 to P, giving us $P = \{12, 21\}$. We also add the rule

$$(1) \rightsquigarrow (12)(21)$$

to our set of rules $\mathcal{R}$.

Now choose 12 from P. First we check to see if 1 is a GT-reducible entry. Since $\|B\|_\infty = 4$, Proposition 4.4.3 shows that we only need to test whether $\ell_r(W(\{132, 3241\}; 2, 2))$ and $\ell_r(T(\{132, 3241\}; 12))$ agree for $r = 1, 2, 3$. These numbers are shown in the following chart.

r	$\ell_r(W(\{132, 3241\}; 2, 2))$	$\ell_r(T(\{132, 3241\}; 12))$
1	2	2
2	5	5
3	13	13

From this chart we may conclude that 1 is a GT-reducible entry, so we remove 12 from P and we replace (12) by (1) in all of our rules. After this we have $P = \{21\}$ and $\mathcal{R}$ contains the single rule

$$(1) \rightsquigarrow (1)(21).$$

We must now choose 21 from P. However, 21 does not have any removable entries, so it is not GT-reducible. We therefore remove 21 from P, add its children (321, 231, and 213) to P, and add the rule

$$(21) \rightsquigarrow (321)(231)(213)$$

to $\mathcal{R}$.

Using the same process as with 12 we can find that both 321 and 231 have GT-reducible entries (the entry 2 is GT-reducible for both permutations), so we replace (321) and (231) in all our rules by (21).

At this point we have $P = \{213\}$. This permutation also has a GT-reducible entry (again, 2) so if we were to follow the pattern of the previous cases we would simply replace all instances of (213) in our set of rules by (12). However, since 12 is GT-reducible itself we instead replace instances of (213) by (1).

We are now done because $P = \emptyset$, and we have found the generating tree

$$\begin{array}{lll} \text{Root:} & (1) & \\ \text{Rule:} & (1) & \rightsquigarrow (1)(21) \\ & (21) & \rightsquigarrow (21)(21)(1). \end{array}$$

Up to relabeling, this is the same tree we found in Proposition 4.2.2. Using the transfer matrix method as we did in Proposition 4.2.4, it is routine to compute the generating function for $\mathrm{Av}(132, 3241)$ from this tree.

4.6 Proof of the main result

It remains to investigate when this procedure terminates. If the nodes of $T(B)$ have arbitrarily large degrees, then we will never reach a state where $P = \emptyset$, because in that case either $12 \cdots n$ or $n \cdots 21$ will never have a removable entry, let alone a GT-reducible entry. Our central result, Theorem 4.6.2 below, shows that if B is finite and $T(B)$ has bounded degrees then this procedure will terminate.

We begin with a technical lemma that will be used to construct large sets of entries satisfying a condition stronger than removability.

Lemma 4.6.1. *Suppose that B is a finite basis containing both a child of an increasing permutation and a child of a decreasing permutation and fix a positive integer r. Every sufficiently long B-avoiding permutation π contains a set $X = \{x_1, \ldots, x_r\}$ of r distinct pairwise nonadjacent entries so that each x_j is removable in the word $\pi - (X \setminus \{x_j\})$.*

Proof. We prove the lemma by induction on r. The base case $r = 1$ is immediate because Theorem 4.3.1 implies that all sufficiently long permutations contain removable entries, and we may take x_1 to be any removable entry in π.

Let π be a B-avoiding permutation of length n. The $r = 2$ case provides a nice illustration of our argument, so we examine it before moving on to the general case. Let $y_1, \ldots, y_s$ denote the removable entries in $\pi - x_1$ that are not adjacent to x_1 in π, and assume to the contrary that x_1 is not removable in $\pi - y_i$ for any $i \in [s]$. Because x_1 is removable in π, at least one of the sites adjacent to x_1 in π is inactive. Now form the B-containing permutation σ by inserting $n+1$ into π in an inactive site adjacent to x_1. Because x_1 is not removable in $\pi - y_i$, $\sigma - y_i$ is B-avoiding. But this means that every copy of a pattern from B in σ must contain the entry y_i and the entry $n+1$, so $s \leq \|B\|_\infty - 1$. On the other hand, Theorem 4.3.1 shows that we may take s to be as large as we like so long as n is sufficiently large, a contradiction.

Now suppose that r is any integer at least 2, and that we have found a set of entries $\{x_1, \ldots, x_{r-1}\}$ satisfying the desired conditions. We wish to find an entry x_r, not adjacent to any of the entries $x_1, \ldots, x_{r-1}$, so that $\{x_1, \ldots, x_r\}$ satisfies the desired conditions. As in the $r = 2$ case before, we begin by letting $y_1, \ldots, y_s$ denote the removable entries of $\pi - x_1 - \cdots - x_{r-1}$ that are not adjacent (in π) to any x_i. Assume to the contrary that none of the y_i's will function adequately as x_r, so for each $i \in [s]$ there is at least one $j \in [r-1]$ such that x_j is not removable in $\pi - x_1 - \cdots - x_{j-1} - x_{j+1} - \cdots - x_{r-1} - y_i$. Choose one of these values to be denoted $j(i)$.

When we were trying to find an entry to serve as x_2 we built a single permutation σ. This time we need to consider the $r-1$ permutations $\sigma_1, \ldots, \sigma_{r-1}$ where σ_j is formed by inserting $n+1$ into π in an inactive site adjacent to x_j and then removing $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{r-1}$. Each of these permtuations contains a pattern from B and by our assumptions, $\sigma_{j(i)} - y_i$ avoids B for each $i \in [s]$. As before, we now ask how many different values of i can share the same value $j(i)$. The answer is the same: every copy of a pattern from B in $\sigma_{j(i)}$ must contain both y_i and $n+1$, so at most $\|B\|_\infty - 1$ values of i may share the same $j(i)$. Therefore we need only have $s > (r-1)(\|B\|_\infty - 1)$, which we get if n is sufficiently large, to guarantee that at least one of the y_i's can serve as x_r, completing the proof of the claim. □

Theorem 4.6.2. *Let B be a finite basis. The pattern-avoidance tree $T(B)$ is isomorphic to a finitely labeled generating tree if and only if B contains both a child of an increasing permutation and a child of a decreasing permutation. Furthermore, if $T(B)$ satisfies these conditions then the algorithm presented in Section 4.5 will find a finitely labeled generating tree isomorphic to $T(B)$.*

Proof. These conditions on B are necessary by Theorem 4.3.1. To prove the other direction, it suffices to show that every sufficiently long permutation is GT-reducible. Take π to be a B-avoiding permutation of length n. We prove the theorem by showing that if r is sufficiently large then at least one of the removable entries $x_1, \ldots, x_r$ guaranteed by Lemma 4.6.1 must be GT-reducible.

If x_j is left-removable in $\pi - (X \setminus \{x_j\})$, set $\epsilon_j = -$. Otherwise x_j must be right-removable in $\pi - (X \setminus \{x_j\})$ and we set $\epsilon_j = +$. Suppose to the contrary that no x_j is GT-reducible, and thus for every $j \in [r]$ there is some $v_j \in W(B; n, \pi - x_j)$ so that $w_j = \iota^{\epsilon_j}_{\pi,x_j}(v_j)$ contains a pattern from B. In fact, Propositions 4.4.1 and 4.4.3 show that we may assume $|w_j| \le n + \|B\|_\infty - 1$.

Each copy of a pattern from B in w_j must use the entry x_j since $\partial_{x_j}(w_j) = v_j$ is B-avoiding. Hence at most $\|B\|_\infty$ different x_j's may share the same w_j. Therefore it would suffice to show that the number of w_j's is bounded by some constant depending only on the set B. To accomplish this we will show that each w_j lies in the set

$$\iota^{\epsilon_1}_{\pi,x_1} \circ \iota^{\epsilon_2}_{\pi-x_1,x_2} \circ \cdots \circ \iota^{\epsilon_r}_{\pi-x_1-\cdots-x_{r-1},x_r}(W(B; n, \pi - X)),$$

by showing that for all $j \in [r]$,

$$\iota^{\epsilon_j}_{\pi,x_j} = \iota^{\epsilon_1}_{\pi,x_1} \circ \iota^{\epsilon_2}_{\pi-x_1,x_2} \circ \cdots \circ \iota^{\epsilon_r}_{\pi-x_1-\cdots-x_{r-1},x_r} \circ \partial_{x_r} \circ \cdots \circ \partial_{x_{j+1}} \circ \partial_{x_{j-1}} \circ \cdots \circ \partial_{x_1}$$

as maps from $W(B; n, \pi - x_j)$ to $W(\emptyset; n, \pi)$. Proposition 4.4.1 implies that this would follow from

$$\iota^{\epsilon_1}_{\pi,x_1} \circ \iota^{\epsilon_2}_{\pi-x_1,x_2} \circ \cdots \circ \iota^{\epsilon_r}_{\pi-x_1-\cdots-x_{r-1},x_r} =$$
$$\iota^{\epsilon_j}_{\pi,x_j} \circ \iota^{\epsilon_1}_{\pi-x_j,x_1} \circ \cdots \circ \iota^{\epsilon_{j-1}}_{\pi-x_1-\cdots-x_{j-2}-x_j,x_{j-1}} \circ \iota^{\epsilon_{j+1}}_{\pi-x_1-\cdots-x_j,x_{j+1}} \circ \cdots \circ \iota^{\epsilon_r}_{\pi-x_1-\cdots-x_{r-1},x_r},$$

and this follows from Observation 4.4.2 because the x_i's are pairwise nonadjacent.

Theorem 4.3.1 gives a bound on the number of children a node in $T(B)$ may have. Let Δ denote this bound. Since $W(B; n, \pi - X) \cong T(B; \mathrm{st}(\pi - X))$, the number of nodes in $W(B; n, \pi - X)$ of height between 0 and $\|B\|_\infty - 1$ is bounded by $1 + \Delta + \Delta^2 + \cdots + \Delta^{\|B\|_\infty - 1}$. Therefore, since we have shown that each w_j lies in the image of this set under the map $\iota^{\epsilon_1}_{\pi,x_1} \circ \iota^{\epsilon_2}_{\pi-x_1,x_2} \circ \cdots \circ \iota^{\epsilon_r}_{\pi-x_1-\cdots-x_{r-1},x_r}$, we have bounded the number of possible w_j's, completing the proof. □

Theorem 4.6.2 only applies when the basis B is finite. This hypothesis is necessary: because Av(321, 4123) contains the infinite antichain U from Section 1.8, Proposition 2.5.1 shows that it contains a subclass which does not even have a holonomic generating function, let alone a finitely labeled generating tree.

4.7 Applications

The algorithm described in Section 4.5 is implemented in the Maple package FinLabel. We have already seen three uses for this package: Propositions 1.5.3 and 1.5.6 and the derivation of (2.6) on page 47. We present here many more examples that can be done with the FinLabel package. First we have results from the classical paper of Simion and Schmidt [130]:

B	generating function for $\mathrm{Av}(B)$
$\{123, 213\}$	$\frac{x}{1-2x}$
$\{123, 231\}$	$\frac{-x(1-x+x^2)}{(x-1)^3}$
$\{123, 321\}$	$x + 2x^2 + 4x^3 + 4x^4$
$\{132, 231\}$	$\frac{x}{1-2x}$
$\{312, 231\}$	$\frac{x}{1-2x}$
$\{123, 132, 213\}$	$\frac{x(1+x)}{1-x-x^2}$
$\{123, 132, 231\}$	$\frac{x}{(x-1)^2}$
$\{123, 132, 321\}$	$x + 2x^2 + 3x^3 + x^4$
$\{123, 231, 312\}$	$\frac{x}{(x-1)^2}$
$\{132, 213, 231\}$	$\frac{x}{(x-1)^2}$

West [159] undertook a systematic study of permutations that avoid one pattern of length three and another of length four. The generating functions that our algorithm can rederive are listed in the chart below.

B	generating function for $\mathrm{Av}(B)$
$\{123, 3214\}$	$\frac{x(1-x)}{1-3x+x^2}$
$\{123, 3241\}$	$\frac{x(1-3x+4x^2-x^3)}{(2x-1)(1-x)^3}$
$\{123, 3421\}$	$\frac{x(1-3x+5x^2-2x^3)}{(1-x)^5}$
$\{123, 4321\}$	$x+2x^2+5x^3+13x^4+25x^5+25x^6$
$\{132, 3214\}$	$\frac{x(1-2x+2x^2)}{1-4x+5x^2-3x^3}$
$\{132, 3241\}$	$\frac{x(1-x)}{1-3x+x^2}$
$\{132, 3421\}$	$\frac{x(1-3x+3x^2)}{(1-x)(2x-1)^2}$
$\{132, 4321\}$	$\frac{x(1-3x+5x^2-2x^3+x^4)}{(1-x)^5}$
$\{213, 1234\}$	$\frac{x(1-x)}{1-3x+x^2}$
$\{213, 1243\}$	$\frac{x(1-x)}{1-3x+x^2}$
$\{213, 1423\}$	$\frac{x(1-x)}{1-3x+x^2}$
$\{213, 4123\}$	$\frac{x(1-x)}{1-3x+x^2}$

Finally we have permutations that avoid two patterns of length four. The following generating functions, recently computed by Kremer and Shiu [92], can also be found using FinLabel.

B	generating function for $\mathrm{Av}(B)$
$\{1234, 3214\}$	$\frac{x(1-3x)}{(x-1)(4x-1)}$
$\{1234, 3241\}$	$\frac{x(1-11x+54x^2-151x^3+268x^4-313x^5+234x^6-108x^7+29x^8-4x^9)}{(1-3x+x^2)(2x-1)^2(x-1)^6}$
$\{1234, 3421\}$	$\frac{x(1-7x+24x^2-44x^3+62x^4-39x^5+32x^6-19x^7+4x^8)}{(1-x)^9}$
$\{1234, 4321\}$	$x+2x^2+6x^3+22x^4+86x^5+306x^6+882x^7+1764x^8+1764x^9$
$\{1243, 3214\}$	$\frac{x(1-4x+5x^2-3x^3)(1-x)}{1-7x+17x^2-22x^3+13x^4-4x^5}$
$\{1243, 3241\}$	$\frac{x(1-9x+31x^2-49x^3+37x^4-14x^5+2x^6)}{(1-x)(1-4x+2x^2)(1-3x+x^2)^2}$
$\{1243, 3421\}$	$\frac{x(1-9x+34x^2-64x^3+64x^4-28x^5+4x^6)}{(x-1)(2x-1)^5}$
$\{1423, 3214\}$	$\frac{x(1-6x+12x^2-7x^3+2x^4)}{1-8x+22x^2-25x^3+10x^4-2x^5}$
$\{1423, 3241\}$	$\frac{x(2x-1)^2(1-x)}{1-7x+16x^2-16x^3+4x^4}$
$\{3214, 4123\}$	$\frac{x(1-3x)}{(x-1)(4x-1)}$

Chapter 5

Enumeration schemes

Notice: The work in this chapter is adapted from Vatter [154].

Zeilberger [166] developed the notion of *enumeration schemes* and wrote the Maple package WILF to automate their discovery. Roughly, enumeration schemes are a divide and conquer technique which aims to partition the class into smaller pieces from which recurrences can be derived.

When enumeration schemes succeed, they provide a polynomial-time algorithm to compute the number of length n permutations in the class. Wilf [160] refers to such algorithms as formulas, but we will call them *Wilfian formulas*.

We review (a symmetry of) Zeilberger's schemes in the next section before extending them considerably in Section 5.2. These extended schemes can be automatically computed with the Maple package WILFPLUS. In Section 5.3 we give several examples of schemes derived using WILFPLUS.

5.1 Zeilberger's original schemes

Take $\pi \in S_k$, suppose that $n \geq k$, and let $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. In Zeilberger's original formalization of enumeration schemes, we divide $\mathrm{Av}_n(B)$ into the sets

$$A_\pi(n; B; i_1, i_2, \ldots, i_k) = \{p \in \mathrm{Av}_n(B) : p(1) = i_{\pi(1)}, \ldots, p(k) = i_{\pi(k)}\}.$$

In words, $A_\pi(n; B; i_1, i_2, \ldots, i_k)$ is the set of B-avoiding length n permutations that begin with the entries $i_1, i_2, \ldots, i_k$, in the order specified by π. For example,

$$A_{312}(9; B; 2, 3, 7) = \{723x_4x_5x_6x_7x_8x_9 \in \mathrm{Av}_n(B)\}. \tag{5.1}$$

In order to make enumeration schemes more closely resemble generating trees and the insertion encoding, we consider a symmetry of his approach. Everywhere Zeilberger mentions a permutation we consider its inverse. Thus we should specify the set of restrictions, B, a set of small entries of some length, π, and the positions in which the entries of π occur. But instead of specifying the positions, we specify the gaps between the entries with a *gap vector*, $\mathbf{g}$. After performing these transformations, our version of (5.1) is

$$Z(B; 231; (1,0,3,2)) = \{x_1 23 x_4 x_5 x_6 1 x_8 x_9 \in \mathrm{Av}_9(B)\},$$

and in general we are concerned with the sets

$$Z(B; \pi; \mathbf{g}) = \{p \in \mathrm{Av}_{k+\|\mathbf{g}\|}(B) : p(g_1 + 1) = \pi(1), \ldots, p(g_1 + \cdots + g_k + k) = \pi(k)\},$$

where k is the length of π and $\|\mathbf{g}\|$ denotes the sum of the components of $\mathbf{g}$. Thus $Z(B; \pi; \mathbf{g})$ is the set of all B-avoiding permutations of length $k + \|\mathbf{g}\|$ whose least k elements occur in the positions $g_1 + 1, g_1 + g_2 + 2, \ldots, g_1 + g_2 + \cdots + g_k + k$ and form a π-subsequence.

Not all pairs $(\pi, \mathbf{g})$ result in a nonempty Z-set. Following Zeilberger, for a length k permutation π we define

$$\mathcal{J}(\pi) = \{j \in [k+1] : Z(B; \pi; \mathbf{g}) = \emptyset \text{ for all } \mathbf{g} \text{ with } g_j > 0\}.$$

Thus $Z(B; \pi; \mathbf{g})$ is guaranteed to be empty if $\mathbf{g}$ does not *obey* $\mathcal{J}(\pi)$, meaning that $g_j \neq 0$ for some $j \in \mathcal{J}(\pi)$.

For example, consider the case $B = \{132\}$. Then $2 \in \mathcal{J}(12)$ because if $g_2 > 0$ then there is some entry between 1 and 2 in every permutation in $Z(B; \pi; \mathbf{g})$, and this gives a 132-pattern. In order to check that $\mathcal{J}(12) = \{2\}$ we need merely observe that 312 and 123 avoid 132. Our approach in this example can easily be generalized to compute $\mathcal{J}(\pi)$ for any π and B.

Proposition 5.1.1. *For any permutation π and basis B, $\mathcal{J}(\pi)$ can be computed by inspecting the B-avoiding children of π.*

Proof. Given i and j, consider the vector $\mathbf{h}$ for which $h_i = 0$ for all $i \neq j$ and $h_j = 1$. If $Z(B; \pi; \mathbf{h}) = \emptyset$ then $Z(B; \pi; \mathbf{g}) = \emptyset$ for all $\mathbf{g}$ with $g_j > 0$, so $j \in \mathcal{J}(\pi)$. If instead $Z(B; \pi; h) \neq \emptyset$ then $j \notin \mathcal{J}(\pi)$. ∎

For any $r \in [k]$, the set $Z(B;\pi;(g_1,\ldots,g_{k+1}))$ embeds naturally (remove the entry $\pi(r)$ and standardize) into

$$Z(B;\mathrm{st}(\pi-\pi(r));(g_1,\ldots,g_{r-1},g_r+g_{r+1},g_{r+2},\ldots,g_{k+1})), \tag{5.2}$$

where $\pi-\pi(r)$ denotes the word obtained from π by omitting the entry $\pi(r)$, so, for example, $51342-1=5342$. To make (5.2) easier to state, we define $d_r(\pi)$ to be $\mathrm{st}(\pi-\pi(r))$ and let

$$d_r((g_1,\ldots,g_{k+1}))=(g_1,\ldots,g_{r-1},g_r+g_{r+1},g_{r+2},\ldots,g_{k+1}).$$

Sometimes the embedding of $Z(B;\pi;\mathbf{g})$ into $Z(B;d_r(\pi);d_r(\mathbf{g}))$ is a bijection. If this is true for all gap vectors $\mathbf{g}$ that obey $\mathcal{J}(\pi)$, that is, that have $g_j=0$ for all $j\in\mathcal{J}(\pi)$, then we say that $\pi(r)$ is *enumeration-scheme-reducible for* π *with respect to* B, or, for short, ES-reducible. (Zeilberger [166] refers to such entries as *reversely deleteable*.) We also say that a permutation with an ES-reducible entry is itself ES-reducible, and a permutation without an ES-reducible entry is ES-irreducible. Note that the GT-reducible elements from the last chapter are also ES-reducible.

For example, suppose again that $B=\{132\}$ and consider the permutation 12. We have already observed that $\mathcal{J}(12)=\{2\}$. Now we claim that the entry 1 is ES-reducible. The gap vectors that obey $\mathcal{J}(12)$ are those of the form $(g_1,0,g_3)$, and thus we would like to verify that the embedding of $Z(\{132\};12;(g_1,0,g_3))$ into $Z(\{132\};1;(g_1,g_3))$ is a bijection.

Take $p\in Z(\{132\};1;(g_1,g_3))$ and consider inverting this embedding. In this case, that amounts to inserting the element 1 into position g_1+1 and increasing all other entries of p by 1. Label the resulting permutation p'. For example, from $\mathbf{g}=(3,0,1)$ and $p=52314$ we obtain $p'=634125$.

We would like to show that p' avoids 132. To show this we consider all possible ways in which the new element 1 could participate in a 132-pattern. Clearly this entry must be the first entry in such a pattern. Now note that the 2 in p' cannot participate in this 132-pattern, because the 1 and 2 are adjacent. But then there is a 132-pattern in p' which uses the 2 instead of the 1, and thus p contains a 132-pattern, a contradiction.

Thus we have shown that

$$|Z_n(\{132\}; 12; (g_1, g_2, g_3))| = \begin{cases} 0 & \text{if } g_2 > 0, \\ |Z_{n-1}(\{132\}; 1; (g_1, g_3))| & \text{if } g_2 = 0. \end{cases}$$

Although this example did not demonstrate it, detecting and verifying ES-reducibility by hand can be enormously tedious. Fortunately, it is also unnecessary. By adapting the proof of Proposition 4.4.3, we arrive at the following test for ES-reducibility that can be routinely checked by computer[1]. In it we let $\|B\|_\infty$ denote the length of the longest permutation in B.

Proposition 5.1.2. *The entry $\pi(r)$ of the permutation π is ES-reducible if and only if*

$$|Z(B; \pi; \mathbf{g})| = |Z(B; d_r(\pi); d_r(\mathbf{g}))|$$

for all gap vectors $\mathbf{g}$ of length $|\pi| + 1$ that obey $J(\pi)$ and satisfy $\|\mathbf{g}\| \leq \|B\|_\infty - 1$.

Proof. If $\pi(r)$ is ES-reducible then the claim follows by definition. To establish the other direction, suppose that $\pi(r)$ is not ES-reducible, and choose $\mathbf{g}$ and $p \in Z(B; d_r(\pi); d_r(\mathbf{g}))$ so that $\mathbf{g}$ obeys $\mathcal{J}(\pi)$ but p cannot be obtained from a permutation in $Z(B; \pi; \mathbf{g})$ by removing $\pi(r)$ and standardizing.

First form the (B-containing) permutation p' by incrementing each entry of p that is at least $\pi(r)$ by 1 and inserting $\pi(r)$ into position $g_1 + \cdots + g_r + r$. Thus p' is the permutation that would have mapped to p, except that p' contains a pattern from B and thus does not lie in $Z(B; \pi; \mathbf{g})$.

Now pick some $\beta \in B$ that is contained in p', and choose a specific occurrence of β in p'. Note that since $p = \mathrm{st}(p' - \pi(r))$ avoids B, this occurrence of β must include the entry $\pi(r)$. Let p'' denote the standardization of the subsequence of p' formed by all entries that are either in the chosen occurrence of β or in π (or in both), so p'' contains a β-pattern and lies in $Z(\emptyset; \pi; \mathbf{h})$ for some $\mathcal{J}(\pi)$-obeying $\mathbf{h}$ with $\|\mathbf{h}\| \leq \|B\|_\infty - 1$. On the other hand, $\mathrm{st}(p'' - \pi(r))$ avoids B, which implies that $|Z(B; d_r(\pi); \mathbf{h})| > |Z(B; \pi; \mathbf{h})|$, as desired. ☐

[1]Zeilberger's approach in [166] used what he referred to as "logical reasoning," and while it is no less rigorous than this approach, Proposition 5.1.2 has the advantage of being very explicit.

For example, consider the basis $B = \{132\}$ again. In order to show that the 1 in $\pi = 12$ is ES-reducible using this proposition, we first find that $\mathcal{J}(12) = \{2\}$ and then perform the following 6 computations:

$\mathbf{g}$	$\lvert Z(\{132\}; 12; \mathbf{g})\rvert$	$\lvert Z(\{132\}; 1; d_1(\mathbf{g}))\rvert$
$(0,0,0)$	1	1
$(0,0,1)$	1	1
$(1,0,0)$	1	1
$(0,0,2)$	1	1
$(1,0,1)$	2	2
$(2,0,0)$	2	2

In a similar manner one can verify that for $B = \{132\}$, $\mathcal{J}(21) = \emptyset$ and the 1 in 21 is ES-reducible. This gives the following enumeration scheme for Av(132):

$$\begin{aligned}
s_n(132) &= |Z(\{132\}; \emptyset; (n))|, \\
|Z(\{132\}; \emptyset; (g_1))| &= \sum_{i=0}^{g_1-1} |Z(\{132\}; 1; (i, g_1 - i - 1))|, && \text{(for } g_1 \geq 1\text{)} \\
|Z(\{132\}; 1; (g_1, g_2))| &= \sum_{i=0}^{g_1-1} |Z(\{132\}; 21; (i, g_1 - i - 1, g_2))| \\
&\quad + \sum_{i=0}^{g_2-1} |Z(\{132\}; 12; (g_1, i, g_2 - i - 1))|, && \text{(for } g_1, g_2 \geq 1\text{)} \\
&= \sum_{i=0}^{g_1} |Z(\{132\}; 1; (i, g_1 + g_2 - i - 1))|. && \text{(for } g_1, g_2 \geq 1\text{)}
\end{aligned}$$

5.2 Extending enumeration schemes

We will replace the sets $\mathcal{J}(\pi)$ in this section, giving us a more powerful version of enumeration schemes that can be found automatically with the Maple package WilfPlus.

In order to motivate this change, we first consider a shortcoming of $\mathcal{J}(\pi)$. Let $B = \{1342, 1432\}$. It can be shown easily, even by hand, that 12 is ES-irreducible. To do so, first note that $\mathcal{J}(12) = \emptyset$, as witnessed by the permutations 312, 132, and 123. Now consider the set $Z(B; 12; (0,2,0))$. This set is empty, but removing 1 gives the nonempty set $Z(B; 1; (2,0))$ while removing 2 gives the nonempty set $Z(B; 1; (0,2))$.

Indeed, this reasoning generalizes to show that all permutations of the form $\ominus^m 12$ are ES-irreducible, so $\mathrm{Av}(1342, 1432)$ does not have a finite enumeration scheme, at least in Zeilberger's original sense.

Zeilberger's enumeration schemes fail in the previous example for a very simple reason: $\mathcal{J}(12)$ is too coarse to capture the fact that $(0, 2, 0)$ is not a valid gap vector for a B-avoiding descendant of 12. We remedy this problem with the following definition.

Definition 5.2.1. *The entry $\pi(r)$ of the length k permutation π is said to be ES$^+$-reducible if*

$$|Z(B; \pi; \mathbf{g})| = |Z(B; d_r(\pi); d_r(\mathbf{g}))| \tag{5.3}$$

whenever $Z(B; \pi; \mathbf{g})$ is nonempty. Further, we say that the permutation π is ES$^+$-reducible if it contains an ES$^+$-reducible entry, and ES$^+$-irreducible otherwise.

We then replace (for now) the set $\mathcal{J}$ by

$$\mathcal{G}(\pi) = \{\mathbf{g} : Z(B; \pi; \mathbf{g}) \neq \emptyset\}.$$

One can view $\mathcal{G}(\pi)$ as a closed class of vectors in $\mathbb{N}^{|\pi|+1}$. Thus we will carry over our Av notation to this context.

Recall that for $\pi(r)$ to be ES-reducible, it had to satisfy (5.3) for all $\mathbf{g}$ that contained 0's in the positions specified by $\mathcal{J}(\pi)$. Clearly if $\mathbf{g} \in \mathcal{G}(\pi)$ then $\mathbf{g}$ obeys $\mathcal{J}(\pi)$, but there can be gap vectors that obey $\mathcal{J}(\pi)$ and do not lie in $\mathcal{G}(\pi)$, as in our previous example with $B = \{1342, 1432\}$. Thus we have obtained a weaker condition by requiring the satisfaction of (5.3) less often.

The proof of Proposition 5.1.2 carries over to this context to give the analogous result on testing for ES$^+$-reducibility.

Proposition 5.2.2. *The entry $\pi(r)$ of the permutation π is ES$^+$-reducible if and only if*

$$|Z(B; \pi; \mathbf{g})| = |Z(B; d_r(\pi); d_r(\mathbf{g}))|$$

for all $\mathbf{g} \in \mathcal{G}(\pi)$ with $\|\mathbf{g}\| \leq \|B\|_\infty - 1$.

For example, let us compute the basis of $\mathcal{G}(12)$ when $B = \{1342, 1432\}$. As already observed, $(0, 2, 0)$ does not lie in $\mathcal{G}(12)$. This gap vector is minimal in $\mathbb{N}^3 \setminus \mathcal{G}(12)$ because

$Z(B;12;(0,1,0))$ is nonempty. To show that the basis of $\mathcal{G}(12)$ is precisely $(0,2,0)$, it suffices to note that the permutation

$$3\ 4\ \cdots\ (g_1+2)\ 1\ (g_1+3)\ 2\ (g_1+4)\ (g_1+5)\ \cdots\ (g_1+g_2+3)$$

avoids B, so $(g_1,1,g_2) \in \mathcal{G}(12)$ for all $g_1, g_2 \in \mathbb{N}$. Thus we have shown that $\mathcal{G}(12) = \text{Av}((0,2,0))$.

Now that we have computed $\mathcal{G}(12)$, it is not hard to check that 2 is ES$^+$-reducible for 12. In order to do so we need to show that the embedding in question is a bijection for all gap vectors (g_1,g_2,g_3) with $g_2 \le 1$.

Suppose to the contrary that the embedding is not onto and take $p \in Z(B;1;(g_1,g_2+g_3))$ that is not mapped to. In other words, the permutation p' obtained from p by inserting 2 into position g_1+g_2+2 and incrementing all the entries of p of value at least 2 contains a 1342 or 1432-pattern. Since p avoids $\{1342,1432\}$, this pattern must involve the entry 2. First, the 2 cannot play the role of the "2" in such a pattern, because then the 1 would be forced to play the role of the "1," and there can be at most one entry between the 1 and 2 since $g_2 \le 1$. The only other possible role for the 2 is as the "1," but in this case we could substitute the 1, thereby finding a B-pattern in p, a contradiction.

We can replace $\mathcal{J}(\pi)$ by $\mathcal{G}(\pi)$ only if we are able to work with $\mathcal{G}(\pi)$. Because $\mathbb{N}^{|\pi|+1}$ is pwo, the basis of $\mathcal{G}(\pi)$ must be finite. However, this basis may be quite large, or may contain vectors with large components. For example, another way to state the Erdős-Szekeres Theorem is that

$$\mathcal{G}(\emptyset) = \text{Av}(((j-1)(k-1)+1))$$

when $B = \{12\cdots j, k\cdots 21\}$. While this does not preclude effective computation of $\mathcal{G}(\pi)$, it does suggest that such computations could be time consuming.

It happens that we can circumvent this problem by replacing $\mathcal{G}(\pi)$ by a different set of gap vectors. First we return to the way in which $\mathcal{G}(\pi)$ is used. By our definitions, if $\pi(r)$ is ES$^+$-reducible for π then

$$|Z(B;\pi;\mathbf{g})| = \begin{cases} 0 & \text{if } \mathbf{g} \notin \mathcal{G}(\pi), \\ |Z(B;d_r(\pi);d_r(\mathbf{g}))| & \text{otherwise.} \end{cases}$$

This equality shows that when enumerating the B-avoiding descendants of π with gap vector $\mathbf{g}$, we first check to see if $\mathbf{g}$ lies in $\mathcal{G}(\pi)$. If $\mathbf{g} \notin \mathcal{G}(\pi)$ then we can be sure that no such descendants exist. If $\mathbf{g} \in \mathcal{G}(\pi)$ and $\pi(r)$ is ES$^+$-reducible, then $Z(B;\pi;\mathbf{g})$ is in one-to-one correspondence with $Z(B;d_r(\pi);d_r(\mathbf{g}))$. Note that if $Z(B;\pi;\mathbf{g})$ and $Z(B;d_r(\pi);d_r(\mathbf{g}))$ are both empty then they are trivially in one-to-one correspondence, so we could instead use the recurrence

$$|Z(B;\pi;\mathbf{g})| = \begin{cases} 0 & \text{if } \mathbf{g} \notin \mathcal{G}(\pi) \text{ and } d_r(\mathbf{g}) \in \mathcal{G}(d_r(\pi)), \\ |Z(B;d_r(\pi);d_r(\mathbf{g}))| & \text{otherwise.} \end{cases}$$

This equality shows that instead of considering $\mathcal{G}(\pi)$, we can look at the larger set of gap vectors for which either $Z(B;\pi;\mathbf{g}) \neq \emptyset$ or $Z(B;d_r(\pi);d_r(\mathbf{g})) = \emptyset$. Unfortunately, this set need not be an ideal[2], so we consider the largest lower order ideal of $\mathbb{N}^{|\pi|+1}$ for which these conditions hold:

$$\mathcal{G}_r(\pi) = \{\mathbf{g} \in \mathbb{N}^{|\pi|+1} : Z(B;\pi;\mathbf{h}) \neq \emptyset \text{ or } Z(B;d_r(\pi);d_r(\mathbf{h})) = \emptyset \text{ for all } \mathbf{h} \leq \mathbf{g}\}.$$

Note that $\mathcal{G}(\pi) \subseteq \mathcal{G}_r(\pi)$: if $\mathbf{g} \in \mathcal{G}(\pi)$ then $Z(B;\pi;\mathbf{g}) \neq \emptyset$, so $Z(B;\pi;\mathbf{h}) \neq \emptyset$ for all $\mathbf{h} \leq \mathbf{g}$, so $\mathbf{g} \in \mathcal{G}_r(\pi)$. With this observation we have

$$|Z(B;\pi;\mathbf{g})| = \begin{cases} 0 & \text{if } \mathbf{g} \notin \mathcal{G}_r(\pi), \\ |Z(B;d_r(\pi);d_r(\mathbf{g}))| & \text{otherwise,} \end{cases}$$

if $\pi(r)$ is ES$^+$-reducible for π and B.

This new set has several advantages over $\mathcal{G}(\pi)$. For one, it may be considerably smaller, thus simplifying the scheme. More importantly, the following bound on basis elements implies that $\mathcal{G}_r(\pi)$ can be found automatically.

Proposition 5.2.3. *Suppose that $\pi(r)$ is an ES$^+$-reducible entry in π (with respect to the permutation class with basis B). Then each basis vector $\mathbf{b}$ of $\mathcal{G}_r(\pi)$ satisfies $\|\mathbf{b}\| \leq \|B\|_\infty - 1$.*

---

[2]An example of this occurs with $B = \{231, k\cdots 21\}$. In Proposition 5.3.1 we observe that the set of $\mathbf{g} \in \mathbb{N}^3$ for which $Z(B;21;\mathbf{g}) \neq \emptyset$ is $\mathrm{Av}((0,1,0),(k-2,0,0))$ while the set of $\mathbf{g} \in \mathbb{N}^2$ for which $Z(B;1;\mathbf{g}) \neq \emptyset$ is $\mathrm{Av}((k-1,0))$, so the set of $\mathbf{g} \in \mathbb{N}^3$ for which either $Z(B;21;\mathbf{g}) \neq \emptyset$ or $Z(B;1;d_1(\mathbf{g})) = \emptyset$ is $\{(g_1,g_2,g_3) : g_1 + g_2 \geq k-1\} \cup \{(g_1,0,g_3) : g_1 \leq k-3\}$.

Proof. Suppose to the contrary that $\mathcal{G}_r(\pi)$ has a basis vector $\mathbf{b}$ with $\|\mathbf{b}\| \geq \|B\|_\infty$. By the minimality of $\mathbf{b}$, we have that $Z(B;\pi;\mathbf{b}) = \emptyset$ and $Z(B;d_r(\pi);d_r(\mathbf{b})) \neq \emptyset$. We also have that $Z(B;\pi;\mathbf{a}) \neq \emptyset$ or $Z(B;d_r(\pi);d_r(\mathbf{a})) = \emptyset$ for every $\mathbf{a} < \mathbf{b}$ because these vectors lie in $\mathcal{G}_r(\pi)$. The latter case can be eliminated easily: because $Z(B;d_r(\pi);d_r(\mathbf{b}))$ is nonempty, $Z(B;d_r(\pi);d_r(\mathbf{a}))$ must also be nonempty for all $\mathbf{a} \leq \mathbf{b}$. Therefore $Z(B;\pi;\mathbf{a}) \neq \emptyset$ for all $\mathbf{a} < \mathbf{b}$.

Now, since $\pi(r)$ is ES$^+$-reducible and $Z(B;\pi;\mathbf{a}) \neq \emptyset$ for all $\mathbf{a} < \mathbf{b}$, we know that there is a bijection between $Z(B;\pi;\mathbf{a})$ and $Z(B;d_r(\pi);d_r(\mathbf{a}))$ for all $\mathbf{a} < \mathbf{b}$. Thus we have the following diagram.

$$\begin{array}{ccc} Z(B;\pi;\mathbf{b}) = \emptyset & & Z(B;d_r(\pi);d_r(\mathbf{b})) \neq \emptyset \\ \downarrow & & \downarrow \\ Z(B;\pi;\mathbf{a}) \neq \emptyset & \longleftrightarrow & Z(B;d_r(\pi);d_r(\mathbf{a})) \neq \emptyset \end{array}$$

The rest of the proof is similar to the proof of Proposition 5.1.2. Choose a permutation $p \in Z(B;d_r(\pi);d_r(\mathbf{b}))$ and form p' by incrementing each entry of p that is at least $\pi(r)$ by 1 and inserting $\pi(r)$ into position $g_1 + \cdots + g_r + r$. Choose a specific occurrence of some $\beta \in B$ in p'. Since p avoids B, this occurrence of β must involve the entry $\pi(r)$. Let p'' denote the standardization of the subsequence of p' given by the entries from π together with the entries from the chosen occurrence of β. Therefore p'' contains a β-pattern and lies in $Z(\emptyset;\pi;\mathbf{a}) \setminus Z(B;\pi;\mathbf{a})$ for some $\mathbf{a} < \mathbf{b}$ with $\|\mathbf{a}\| \leq \|B\|_\infty - 1$. However, this is a contradiction because $d_r(p'')$ avoids B and thus lies in $Z(B;d_r(\pi);d_r(\mathbf{a}))$, and we have assumed that d_r is a bijection between $Z(B;\pi;\mathbf{a})$ and $Z(B;d_r(\pi);d_r(\mathbf{a}))$. □

We conclude this section by writing out the enumeration scheme that we have derived for Av(1342, 1432).

$$
\begin{aligned}
s_n(1342,1432) &= |Z(\{1342,1432\};\emptyset;(n))|,\\
|Z(\{1342,1432\};\emptyset;(g_1))| &= \sum_{i=0}^{g_1-1} |Z(\{1342,1432\};1;(i,g_1-i-1))|, && \text{(for } g_1 \geq 1)\\
|Z(\{1342,1432\};1;(g_1,g_2))| &= \sum_{i=0}^{g_1-1} |Z(\{1342,1432\};21;(i,g_1-i-1,g_2))|\\
&\quad + \sum_{i=0}^{g_2-1} |Z(\{1342,1432\};12;(g_1,i,g_2-i-1))|, && \text{(for } g_1,g_2 \geq 1)\\
&= \sum_{i=0}^{g_1-1} |Z(\{1342,1432\};1;(i,g_1+g_2-i-1))|\\
&\quad + 2|Z(\{1342,1432\};1;(g_1,g_2-1))|. && \text{(for } g_1,g_2 \geq 1)
\end{aligned}
$$

5.3 Applications

Here we present several examples of finite enumeration schemes. In these presentations, we adopt the following pictorial representation. If π is ES$^+$-irreducible, so $|Z(B;\pi;\mathbf{g})|$ is computed by summing over the B-avoiding children of π, then we draw a solid arrow from π to each of its children. If the entry $\pi(r)$ if ES$^+$-reducible in π then we draw a dashed arrow from π to $d_r(\pi)$, label this arrow with d_r, and indicate the basis of $G_r(\pi)$ beneath π. For example, the enumeration scheme for Av$(1342, 1432)$ is shown in Figure 5.1. We also define the *depth* of an enumeration scheme to be the least integer k so that every permutation of length at least k is ES$^+$-reducible.

As can be seen in Figure 5.1, we include the empty permutation, $\emptyset$, in our diagrams. Although this rarely has no more effect than making our diagrams consume more vertical space, there are a few exceptions. One is the set of all permutations, Av$(\emptyset)$, which has the enumeration shown in Figure 5.2. Another exception is Av$(M_{a,b})$ where $M_{a,b} = \{\beta \in S_{a+b+1} : \beta(a+1) = 1\}$. This class was first counted by Mansour [99]. Its enumeration scheme is shown in Figure 5.3.

The scheme for Av(1234), which thereby produces the sequence (1.2) on page 11 is shown in Figure 5.4.

The 321-hexagon-avoiding permutations, which can be defined as

$$\mathrm{Av}(321, 46718235, 46781235, 56718234, 56781234),$$

have the scheme shown in Figure 5.5. This class was first introduced by Billey and Warrington [26], who showed how to compute the Kazhdan-Lusztig polynomials for them. Stankova and West [137] proved that the number, s_n, of these permutations of length n satisfies

$$s_n = 6s_{n-1} - 11s_{n-2} + 9s_{n-3} - 4s_{n-4} - 4s_{n-5} + s_{n-6}$$

for all $n \geq 7$, which gives sequence A058094 in the OEIS [131]. (So this complicated scheme produces only a sequence with a rational generating function.) Later, Mansour and Stankova [101] counted 321-($2k$)-gon-avoiding permutations for all k.

The *freely braided permutations* are the class

$$\mathrm{Av}(3421, 4231, 4312, 4321).$$

This class was introduced by Green and Losonczy [76] and also arises in the work of Tenner [151]. Mansour [100] found that the freely braided permutations have the generating function

$$\frac{1 - 3x - 2x^2 + (1+x)\sqrt{1-4x}}{1 - 4x - x^2 + (1-x^2)\sqrt{1-4x}}.$$

(Sequence A108600 in the OEIS [131].) They also have an enumeration scheme of depth 3, shown in Figure 5.6.

Chow and West [44] showed, using generating trees, that the classes in Figure 5.7 are Wilf-equivalent, that their generating functions are rational[3], and that these generating functions can be expressed as quotients of Chebyshev polynomials. Krattenthaler [90] gave another proof via a bijection to Dyck paths (in fact, he found the bivariate generating functions for 231-avoiding permutations by length and number of copies of $12 \cdots k$ and for 321-avoiding permutations by length and number of copies of $23 \cdots k1$).

---

[3]This fact can be verified quite quickly: by inverting the permutations in $\mathrm{Av}(321, 23 \cdots k1)$ one obtains the class $\mathrm{Av}(321, k12 \cdots (k-1))$, which satisfies the hypotheses of Theorem 4.6.2, and thus also of Theorem 6.2.1. Note, however, that while symmetries of these classes always have finitely labeled generating trees, and thus also regular insertion encodings, the complexity of their generating trees and insertion encodings increases with k, while the complexity of their enumeration schemes stay fixed.

Around the same time as that work, several other authors studied these and similar classes: Jani and Rieper [83], Mansour and Vainshtein [102, 103], and Robertson, Wilf, and Zeilberger [125]. Deutsch, Hildebrand, and Wilf [51] used these results in their study of the longest increasing subsequence problem for 132-avoiding permutations.

For any fixed k, WilfPlus can automatically and rigorously derive enumeration schemes for these classes. This makes it easy to make conjectures for the general form of these enumeration schemes for all k, although they must be verified by hand. We carry this out for one of these classes below.

Proposition 5.3.1. *The enumeration scheme for* $\mathrm{Av}(231, k \cdots 21)$ *is as shown in Figure 5.7 (b).*

Proof. Let $B = \{231, k \cdots 21\}$. First we verify the computations of $G_r(\pi)$ given in the diagram. For 12, the diagram shows that $\mathcal{G}_1(12) = \mathbb{N}^3$. In any permutation in $Z(B; 12; (g_1, g_2, g_3))$, the entries before the 2 (other than the 1) must be in decreasing order from left to right as any ascent before the 2 would give rise to a 231-pattern. Then, since these permutations must avoid $k \cdots 21$, there can be at most $k-2$ new entries before the 2, so $(g_1, g_2, g_3) \notin \mathcal{G}(12)$ whenever $g_1 + g_2 \geq k-1$. The B-avoiding permutations

$$g_1+g_2+2, \ldots, g_2+4, g_2+3, 1, g_2+2, \ldots, 4, 3, 2, g_1+g_2+3, g_1+g_2+4, \ldots, g_1+g_2+g_3+2$$

then show that $\mathcal{G}(12)$ is $\mathrm{Av}(\{(g_1, g_2, g_3) : g_1 + g_2 \geq k-1\})$. It can be shown similarly that $\mathcal{G}(1) = \mathrm{Av}((k-1, 0))$. This shows that $\mathcal{G}_1(12) = \mathbb{N}^3$ because $Z(B; 1; d_1(\mathbf{g}))$ is empty whenever $Z(B; 12; \mathbf{g})$ is empty.

The computation for 21 is similar. One first checks that

$$\mathcal{G}(21) = \mathrm{Av}((0, 1, 0), (k-2, 0, 0)).$$

Since $\mathcal{G}(1) = \mathrm{Av}((k-1, 0))$, the set of gap vectors $\mathbf{g}$ for which either $Z(B; 21; \mathbf{g}) \neq \emptyset$ or $Z(B; 1; d_1(\mathbf{g})) = \emptyset$ is $\{(g_1, g_2, g_3) : g_1 + g_2 \geq k-1\} \cup \{(g_1, 0, g_3) : g_1 \leq k-3\}$. This implies that $\mathcal{G}_1(21) = \mathrm{Av}((0, 1, 0), (k-2, 0, 0))$. Now we must verify the ES$^+$-reducible entries. The 12 case is clear because the only way 1 can participate in a 231 or $k \cdots 21$-pattern is as the last entry, and in either case the 2 could play the same role. For 21, first note that the 2 also cannot participate in any 231 or $k \cdots 21$-pattern if $\mathbf{g} \in \mathcal{G}_1(21)$: it cannot be the "2" in a 231 because there can be nothing between the 2 and the 1, and it cannot be the "2" in a $k \cdots 21$-pattern because our restrictions on $\mathbf{g}$ prevent there from being enough entries to the left of the 2 to accommodate such a pattern. This shows that the 2 can only possibly be the minimal entry in a forbidden pattern, but then the 1 could play the same role. □

As demonstrated by these examples, another interesting question is the equivalence problem:

Question 5.3.2. *Is it decidable whether two finite enumeration schemes produce the same sequence?*

Note that Question 5.3.2 would be answered by an affirmative resolution of Question 2.2.5. Brlek, Duchi, Pergola, and Rinaldi [40] consider the equivalence problem for generating trees with infinitely many labels.

In addition to the examples already mentioned, there are several other interesting classes avoiding a pair of permutations of length four:

1. Kremer and Shiu [92] proved that there are four classes of this form counted by $(4^{n-1} + 2)/3$. These all have finite enumeration schemes: $\text{Av}(1234, 2143)$ and $\text{Av}(1432, 2341)$ have schemes of depth 3, $\text{Av}(2341, 4321)$ has a scheme of depth 4, and $\text{Av}(2143, 4123)$ has a scheme of depth 6. (The second and third classes mentioned can also be counted with FinLabel.)

2. Kremer [91] (see also Stanley [145, Exercise 6.39.l]) completed the characterization of the classes defined by avoiding two permutations of length four that are counted by the large Schröder numbers. Up to symmetry, there are ten such classes, seven of which have finite enumeration schemes:

Class	Finite enumeration scheme?
Av(1342, 2341)	Yes, of depth 3
Av(1342, 1432)	Yes, of depth 2, shown in Figure 5.1
Av(2341, 2413)	No
Av(2413, 3142)	No, these are the separable permutations considered in Proposition 6.3.2
Av(2431, 3241)	No
Av(3241, 3421)	Yes, of depth 4
Av(3241, 4231)	Yes, of depth 2 (Knuth [89] proved that these are precisely the permutations that can be sorted by an input-restricted deque)
Av(3412, 3421)	Yes, of depth 3
Av(3421, 4321)	Yes, of depth 2
Av(3421, 4231)	Yes, of depth 4

3. The skew-merged permutations from Section 1.7.1 have an enumeration scheme of depth four.

Chapter 6

The four systematic approaches to enumeration

Notice: The work in this chapter is adapted from Vatter [154].

6.1 Summary & comparison

The enumeration of permutation classes, whose ancestry can be traced back to at least 1915 (MacMahon [98]), has frequently been accomplished by beautiful arguments utilizing such diverse objects as Young tableaux, Dyck paths, and planar maps, to name only a few. Our concern here has instead been with systematic methods for solving the enumeration problem. Let us adopt a strict definition of systematic, insisting that the computations can be performed without any human interaction whatsoever. For the definition of enumeration, let us insist on a Wilfian formula (to repeat, a polynomial time (in n) algorithm to compute the number of length n permutations in the class). We have so far seen three general methods that meet these requirements, and we introduce one more in the next section, bringing the list to:

- substitution decompositions (Section 3.2.2),
- generating trees (Chapter 4),
- enumeration schemes (Chapter 5),
- the insertion encoding (Section 6.2).

Each of the four systematic approaches for permutation class enumeration has a natural notion of a "state," and in each case if the class is such that only finitely many states are needed then — at least in principle — these methods give a Wilfian formula for the number of length n permutations in the class. For generating trees, the states are the labels of the isomorphic generating tree. The classes possessing a generating tree with only finitely many labels are characterized by Theorem 4.6.2, and their automatic enumeration can be carried out with the FinLabel package. For the insertion encoding, which associates a language to the permutation class, the natural notion of "state" is a state in the accepting automaton for the associated language. The classes that require only finitely many states (or in other words, the classes that correspond to regular languages) were characterized by Albert, Linton, and Ruškuc [7]; their result appears here as Theorem 6.2.1. For enumeration schemes the translation of "state" is ES^+-irreducible permutation (or, for Zeilberger's original schemes, ES-irreducible permutation). Should a class contain only finitely many such permutations then WilfPlus can automatically enumerate it. No characterization of these classes is known[1]. Moreover, unlike the other methods, there are subclasses of classes with finite enumeration schemes which do not themselves have finite enumeration schemes[2], indicating that such a characterization may be too much to hope for. For substitution decompositions, simple permutations play the role of states. As with enumeration schemes, there is no known characterization of the classes that contain only finitely many simple permutations.

The classes that these techniques can automatically enumerate are related as shown in Figure 6.1, which is to say, they are not very closely related at all. We have already remarked in Section 5.1 that (finitely based) classes with finitely labeled generating trees have finite enumeration schemes, and we will observe that they also give rise to regular insertion encodings in the next section. The lack of other containments in this diagram is established via examples in Section 6.3.

Care should be taken when reading one symbol in this diagram; while the inclusion from finitely labeled generating trees to finite enumeration schemes indicates an increase in the number of classes that can be counted, there is a corresponding decrease in information. Finitely labeled generating trees and regular insertion encodings show

---

[1]Zeilberger [166] dismisses this by stating "if we know beforehand that we are guaranteed to succeed, then it is not research, but doing chores."

[2]In fact, the set of all permutations, Av($\emptyset$), has a finite enumeration scheme (shown in Figure 5.2 on page 97), while several examples of classes without finite enumeration schemes were given in Section 6.3.

that a class has a rational generating function, while classes with only finitely many simple permutations have algebraic generating functions. It is not yet known what types of generating functions can arise from finite enumeration schemes, but they need not be algebraic. For example, Av(1234), which has a holonomic but non-algebraic generating function, has a finite enumeration scheme[3] (shown in Figure 5.4 on page 98). It is natural to hope that finite enumeration schemes produce only holonomic sequences, but this hope remains unproved.

Question 6.1.1. *Is every sequence produced by a finite enumeration scheme holonomic?*

Perhaps the greatest loss of information occurs with Wilf-equivalence (although this would not be an issue if the answer to Question 2.2.5 is "yes"). If two classes both have finitely labeled generating trees, regular insertion encodings, or finitely many simple permutations then, since we can compute their generating functions from this information, we can decide whether or not they are Wilf-equivalent. For enumeration schemes this issue remains unknown (see Question 5.3.2). Occasionally, as with the enumeration schemes pictured in Figure 5.7 (a) and (b) on page 101, the Wilf-equivalence of two classes can be easily deduced from their enumeration schemes, but we also showed several examples in Section 5.3 where such deductions do not readily present themselves.

As the examples in Section 6.3 will demonstrate, there remain considerable differences in the applicability of enumeration schemes, substitution decompositions, and the insertion encoding. This obviously suggests the following question.

Question 6.1.2. *Is there a systematic method of permutation class enumeration which is applicable to all classes with finite enumeration schemes, all classes with regular insertion encodings, and all classes with only finitely many simple permutations?*

Additionally, one would like such a method to be invariant under the eight permutation class symmetries, but Question 6.1.2 is probably demanding enough in the form above.

[3] A more trivial example would be the class of all permutations.

6.2 The insertion encoding

The insertion encoding, recently introduced by Albert, Linton, and Ruškuc [7], is a correspondence between permutation classes and languages. With it, one may attack the enumeration problem with all the tools of formal language theory. Roughly, this correspondence associates to each permutation a word describing how that permutation evolved. At each stage until the desired permutation has been constructed, at least one open *slot* (represented by a $\diamond$) exists in the intermediate *configuration*, and to proceed to the next configuration we insert a new maximal entry into one of these slots. This insertion can occur in four possible ways:

- the slot can be filled (replacing a $\diamond$ by n),
- the new entry can be inserted to the left of the slot (replacing a $\diamond$ by $n\,\diamond$),
- the new entry can be inserted to the right (replacing a $\diamond$ by $\diamond\, n$), or
- the slot can be divided into two slots with the new entry in between (replacing a $\diamond$ by $\diamond\, n\,\diamond$).

These operations are denoted by the symbols $\mathbf{f}$, $\mathbf{l}$, $\mathbf{r}$, and $\mathbf{m}$, respectively. Since each of these operations can be performed on any open slot at any stage, we subscript their symbols with the number of the slot they were applied to (read from left to right). For example, the permutation 31254 has the insertion encoding $\mathbf{m}_1\mathbf{l}_2\mathbf{f}_1\mathbf{r}_1\mathbf{f}_1$ because its evolution is

$$\diamond$$

$$\diamond\,1\,\diamond$$

$$\diamond\,12\,\diamond$$

$$312\,\diamond$$

$$312\ \diamond\ 4$$

$$31254$$

Let $\mathcal{SB}(k)$ denote the permutation class whose basis consists of all length $2k+1$ permutations of the form $babab\cdots bab$ where the a's represent the elements $\{1, 2, \ldots, k\}$ and the b's represent the elements $\{k+1, k+2, \ldots, 2k+1\}$. These classes are called *slot bounded* because in the evolution of a permutation in $\mathcal{SB}(k)$ there are never more than $k+1$ open slots.

Theorem 6.2.1 (Albert, Linton, and Ruškuc [7])**.** *The insertion encoding of a finitely based class is regular if and only if the class is a subclass of $\mathcal{SB}(k)$ for some k.*

One can show using the Erdős-Szekeres Theorem (or one can refer to the proof in [7]) that Theorem 6.2.1 includes all of the classes identified by Theorem 4.6.2 as having finitely labeled generating trees.

Even when the insertion encoding of a class is not regular, useful information can still be obtained by this correspondence. For example, in Section 1.4.3 we mentioned that Albert, Elder, Rechnitzer, Westcott, Zabrocki [1] proved $s_n(1324) > 9.35^n$ for sufficiently large n, thereby disproving a conjecture of Arratia [10]. They did so by considering regular approximations to the insertion encoding of Av(1324). Additionally, Albert, Linton, and Ruškuc [7] consider several classes with context-free insertion encodings and are able to obtain their (algebraic) generating functions from these languages. However, the derivation of insertion encodings is only automatic for subclasses of $\mathcal{SB}(k)$, and thus we choose to limit our focus to this case.

6.3 Non-applications

In this section we collect numerous negative results needed to justify the lack of inclusions in Figure 6.1. The class Av(123) shows that classes can have finite enumeration schemes without having regular insertion encodings, since it has a finite enumeration scheme and a irrational generating function. An example of a class with a regular insertion encoding but without a finite enumeration scheme is given by Av(1234, 4231). It can be computed that this classes lies in $\mathcal{SB}(4)$, but the following proposition shows that it does not have a finite enumeration scheme.

Proposition 6.3.1. *For all k, the permutation $k\cdots 21$ is ES$^+$-irreducible for* Av(1234, 4231).

Proof. Let $\pi = k\cdots 21$. First we show that the entries $\pi(r) = k-r+1$ for $r \in [k-1]$ are ES$^+$-irreducible. Let $\mathbf{g}$ denote the vector in $\mathbb{N}^{k+1}$ which is identically zero except for g_r

and g_{r+1}, which are both 1 (these two components correspond to the gaps on either side of $\pi(r)$). Now observe that there are two permutations in $Z(\{1234, 4231\}; d_r(\pi); d_r(\mathbf{g}))$:

$$k(k-1)\cdots(k-r+2)(k+1)(k+2)(k-r)\cdots 21, \quad \text{and}$$
$$k(k-1)\cdots(k-r+2)(k+2)(k+1)(k-r)\cdots 21,$$

while $Z(\{1234, 4231\}; \pi; \mathbf{g})$ contains only one permutation,

$$k(k-1)\cdots(k-r+2)(k+1)(k-r+1)(k+2)(k-r)\cdots 21.$$

Thus $\mathbf{g} \in \mathcal{G}_r(\pi)$ but $Z(\{1234, 4231\}; \pi; \mathbf{g})$ and $Z(\{1234, 4231\}; d_r(\pi); d_r(\mathbf{g}))$ are not in one-to-one correspondence, so $\pi(r)$ is not ES$^+$-reducible for any $r \in [k-1]$.

To show that $\pi(k) = 1$ is not ES$^+$-reducible, it suffices to consider the gap vector $\mathbf{g} = (0, 0, \ldots, 0, 3, 0)$, for which $|Z(\{1234, 4231\}; \pi; \mathbf{g})| < |Z(\{1234, 4231\}; d_k(\pi); d_k(\mathbf{g}))|$, finishing the proof. □

The substitution decomposition approach appears, at least on the surface, completely independent from the other three methods. The class Av(321, 2341, 3412, 4123) has a finitely labeled generating tree by Theorem 4.6.2 (and thus it also has a finite enumeration scheme and a regular insertion encoding), but we showed in Section 1.5.5 that this class is the set of all permutations that embed into the increasing oscillating sequence, and thus it contains infinitely many simple permutations.

Moreover, the separable permutations from Section 1.5.4, which contain only three simple permutations, possess neither a finite enumeration scheme nor a regular insertion encoding: to see that they do not have a regular insertion encoding, we need only note that they have an irrational generating function (so, for this purpose, Av(231) would function just as well).

To establish that this class does not have a finite enumeration scheme, thereby completing our list of negative examples, we show that every permutation which consists of an increasing sequence followed by a decreasing sequence is ES$^+$-irreducible for Av(2413, 3142). (This set is the W-class $W(1, -1)$, which has basis $\{213, 312\}$ and contains 2^{n-1} permutations of length n.) Therefore, not only do the separable permutations from Subsection 1.5.4 not have a finite enumeration scheme, but for each n

there are 2^{n-1} ES$^+$-irreducible permutations. Moreover, one can observe (either from the definition or the basis) that this class is invariant under the eight permutation class symmetries, so none of these offer any simplification.

Proposition 6.3.2. *Every* $\pi \in W(1,-1)$ *is ES*$^+$*-irreducible for* $\mathrm{Av}(2413, 3142)$.

Proof. Take a length k permutation $\pi \in W(1,-1)$. First we show that $\pi(r)$ is not ES$^+$-reducible for any $2 \leq r \leq k$. If $\pi(r-1) > \pi(r)$, consider the gap vector $\mathbf{g}$ with all components 0 except for $g_r = g_{r+1} = 1$. Then $Z(\{2413, 3142\}; \pi; \mathbf{g})$ contains exactly one permutation, because to avoid 2413, the smaller of these new entries must be to the left of the larger one. However, $Z(\{2413, 3142\}; d_r(\pi); d_r(\mathbf{g}))$ contains 2 permutations (the most possible). The case where $\pi(r-1) < \pi(r)$ can be handled similarly with the gap vector that is 0 except for $g_{r-1} = g_r = 1$. Finally, $\pi(1)$ is not ES$^+$-reducible because $Z(\{2413, 3142\}; \pi; (2, 1, 0, \dots, 0))$ contains at most 5 permutations (the new entries cannot form a 132-pattern, because that would give rise to a 2413-pattern), while $Z(\{2413, 3142\}; d_1(\pi); (3, 0, \dots, 0))$ contains 6 permutations. ☐

Chapter 7

Grid classes

Notice: The work in this chapter is joint with Huczynska.

7.1 Griddability

Important note: In this chapter we index matrices beginning from the lower left-hand corner and we reverse the rows and columns; for example $M_{3,2}$ will denote for us the entry of M in the 3rd column from the left and 2nd row from the bottom. Below we include a matrix with its entries labeled:

$$\begin{pmatrix} (1,2) & (2,2) & (3,2) \\ (1,1) & (2,1) & (3,1) \end{pmatrix}.$$

We have already introduced the best-studied grid class, the skew-merged permutations from Subsection 1.7.1. We defined them there as the permutations that can be written as the union of an increasing subsequence and a decreasing subsequence, and Theorem 1.7.1 states that their basis is $\{2143, 3412\}$.

Roughly, the *grid class* of a matrix M is the set of all permutations that can be divided in a prescribed manner (dictated by M) into a finite number of monotonic blocks. For example, skew-merged permutations can be divided into four monotonic blocks, two increasing and two decreasing, as indicated in Figure 7.1.

As a grid class, the skew-merged permutations are denoted

$$\text{Grid}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

but before reaching that point we need to introduce some notation.

Take a permutation $\pi \in S_n$ and sets $A, B \subseteq [n]$. By $\pi(A \times B)$ we mean the subsequence of π (we will sometimes call this subsequence a *block*) with indices from A which has values in B. For example, applying this operation to the permutation shown in Figure 7.1, we get

$$917456328([5] \times [5]) = 1, 4, 5,$$

and this (increasing) subsequence represents the points in the lower left-hand box of Figure 7.1. The (increasing) subsequence in the upper right-hand box is

$$917456328([6,9] \times [6,9]) = 6, 8,$$

while the (decreasing) subsequence in the lower right-hand box is

$$917456328([6,9] \times [5]) = 3, 2.$$

Now suppose that M is a $t \times u$ matrix (meaning, in the notation of this chapter, that it has t columns and u rows). An *M-gridding* of the permutation $\pi \in S_n$ is a pair of sequences $1 = c_1 \leq c_2 \leq \cdots \leq c_{t+1} = n+1$ (the column divisions) and $1 = r_1 \leq r_2 \leq \cdots \leq r_{u+1} = n+1$ (the row divisions) such that for all $k \in [t]$ and $\ell \in [u]$, $\pi([c_k, c_{k+1}) \times [r_\ell, r_{\ell+1}))$ is:

- increasing if $M_{k,\ell} = 1$,
- decreasing if $M_{k,\ell} = -1$,
- empty if $M_{k,\ell} = 0$.

We define the *grid class of M*, written $\mathrm{Grid}(M)$, to be the set of all permutations that possess an M-gridding. We say that π is *$t \times u$-griddable* if it is M-griddable for some $t \times u$ matrix M.

A class $\mathcal{C}$ is said to be *$t \times u$-griddable* if every permutation in $\mathcal{C}$ is $t \times u$-griddable, and it is said to be *griddable* if it is $t \times u$-griddable for some $t, u \in \mathbb{N}$. Note that all griddable classes lie in some particular grid class (suppose that $\mathcal{C}$ is $t \times u$ griddable and take a larger matrix M containing every $t \times u$ matrix, then $\mathcal{C}$ lies in $\mathrm{Grid}(M)$).

One example of grid classes are the profile classes of Atkinson [12][1], which in our language are grid classes of permutation matrices. Atkinson proved that these classes are enumerated by polynomials (this is a special case of Theorem 7.3.4) and used this fact to solve several enumeration problems. For example, he showed that

$$\operatorname{Av}(132, 4321) = \operatorname{Grid}\left(M^{32415}\right) \cup \operatorname{Grid}\left(M^{42135}\right),$$

from which he computed

$$s_n(132, 4321) = 2\binom{n}{4} + \binom{n+1}{3} + 1.$$

Another example of grid classes are the W-classes from Subsection 1.7.2; these are the grid classes of row vectors. Atkinson, Murphy, and Ruškuc [13] proved that all W-classes are pwo (Theorem 1.7.2). This fact is not true for grid classes in general. For example, the antichain W from Section 1.8 is skew-merged. Such a gridding for w_3 is shown in Figure 7.2.

For any $t \times u$ matrix M we construct a bipartite graph $G(M)$ with vertices x_1, x_2, $\ldots$, x_t and y_1, y_2, $\ldots$, y_u and edges $x_k y_\ell$ precisely when $M_{k,\ell} \neq 0$. For example, the bipartite graph of a vector is a star together with isolated vertices, while the bipartite graph of $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ is C_4. As shown by the following theorem, the pwo properties of a grid class depend only on its graph.

Theorem 7.1.1 (Murphy and Vatter [113])**.** *The grid class of M is pwo if and only if $G(M)$ is a forest.*

It is very tempting to speculate that the enumerative properties of grid class also depend only on its graph. We will show in Theorem 7.3.4 that when $G(M)$ is a matching[2] then all subclasses of $\operatorname{Grid}(M)$ have eventually polynomial enumeration, while Theorem 1.7.3 shows that all subclasses of $\operatorname{Grid}(M)$ have rational generating functions when $G(M)$ is a star. We conjecture that this latter result extends to forests.

[1]It is for this reason that when grid classes were first introduced in Murphy and Vatter [113] they were called profile classes. This turned out to be unfortunate for a number of reasons, and so we are now attempting to rectify matters by giving them this new, better, name.

[2]A *matching* is a graph with no incident edges, or equivalently a graph with maximum degree equal to 1.

Conjecture 7.1.2. *If $G(M)$ is a forest then all subclasses of* Grid(M) *have rational generating functions.*

When $G(M)$ contains a cycle it also contains an infinite antichain, so Proposition 2.5.1 shows that we must be more careful.

Question 7.1.3. *Do all finitely based subclasses of a grid class have algebraic generating functions?*

7.2 The characterization of griddable classes

It appears surprisingly difficult to compute the basis of Grid(M) when M is neither a vector nor a permutation matrix. In fact, only two results are known: the basis for the skew-merged permutations already discussed, and the basis for Grid $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, which has recently been computed by Waton [private communication]. In particular, the following remains a conjecture.

Conjecture 7.2.1. *All grid classes are finitely based.*

We will instead take a coarser approach here and only ask for a characterization of the griddable classes, that is, the permutation classes that lie in some grid class.

It will be useful to have an alternative interpretation of griddability. We say that the permutation $\pi \in S_n$ can be *covered by s monotonic rectangles* if there are $[w_1, x_1] \times [y_1, z_1]$,...,$[w_s, x_s] \times [y_s, z_s] \subseteq [n] \times [n]$ such that

- for each $i \in [s]$, $\pi([w_i, x_i] \times [y_i, z_i])$ is monotone, and
- $\displaystyle\bigcup_{i\in[s]} [w_i, x_i] \times [y_i, z_i] = [n] \times [n]$.

Note that we allow these rectangles to intersect. By definition every $t \times u$-griddable permutation can be covered by tu monotonic rectangles. The other direction of this connection is established by the following proposition.

Proposition 7.2.2. *Every permutation that may be covered by s monotonic rectangles is $2s - 1 \times 2s - 1$-griddable.*

Proof. Suppose that $\pi \in S_n$ is covered by the s monotonic rectangles $[w_1, x_1] \times [y_1, z_1]$, $\ldots$, $[w_s, x_s] \times [y_s, z_s] \subseteq [n] \times [n]$. Define the indices $c_1, c_2, \ldots, c_{2s}$ and $r_1, r_2, \ldots, r_{2s}$ by

$$\begin{aligned}
\{c_1 \leq c_2 \leq \cdots \leq c_{2s}\} &= \{w_1, x_1, w_2, x_2, \ldots, w_s, x_s\}, \\
\{r_1 \leq r_2 \leq \cdots \leq r_{2s}\} &= \{y_1, z_1, y_2, z_2, \ldots, y_s, z_s\}.
\end{aligned}$$

Since these rectangles cover π, we must have $c_1 = r_1 = 1$ and $c_{2s} = r_{2s} = n$. Now we claim that these sets of indices form an M-gridding of π for some $2s - 1 \times 2s - 1$ matrix M.

To prove this claim it suffices to show that $\pi([c_k, c_{k+1}] \times [r_\ell, r_{\ell+1}])$ is monotone for every $k, \ell \in [2s - 1]$, since we can then construct the matrix M based on whether this subsequence is increasing or decreasing. Because the rectangles given cover π, the point (c_k, r_ℓ) lies in at least one rectangle, say $[w_m, x_m] \times [y_m, z_m]$. Thus $c_k \geq w_m$ and $r_\ell \geq y_m$ and, because of the ordering of the c's and r's, we have $c_{k+1} \leq x_m$ and $r_{\ell+1} \leq z_m$. Therefore $[c_k, c_{k+1}] \times [r_\ell, r_{\ell+1}]$ is contained in $[w_m, x_m] \times [y_m, z_m]$ and so $\pi([c_k, c_{k+1}] \times [r_\ell, r_{\ell+1}])$ is monotone. $\square$

With this new interpretation of griddability, we can now characterize the griddable classes.

Theorem 7.2.3. *A permutation class is griddable if and only if it does not contain arbitrarily long sums of* 21 *or skew sums of* 12*.*

Proof. If a permutation class does contain arbitrarily long sums of 21 or skew sums of 12, then it is clearly not griddable.

For the other direction, let $\mathcal{C}$ be a permutation class that does not contain $\ominus^{a+1}12$ or $\oplus^{b+1}21$. We show by induction on $a + b$ that there is a function $f(a, b)$ so that every permutation in $\mathcal{C}$ can be covered by $f(a, b)$ monotonic rectangles, and thus we will be done by Proposition 7.2.2.

First note that if either a or b is 0 then $\mathcal{C}$ can only contain monotone permutations, so we can set $f(a, 0) = f(0, b) = 1$. The next case is $a + b = 2$, and since we may assume that $a, b \neq 0$, we have $a = b = 1$. Thus $\mathcal{C}$ contains neither $\ominus^2 12 = 3412$ nor $\oplus^2 21 = 2143$, so $\mathcal{C}$ is a subclass of the skew-merged permutations and thus every permutation in $\mathcal{C}$ may be covered by 4 monotonic rectangles and we may take $f(1,1) = 4$.

Now, by symmetry and the cases we have already handled, we may assume that $a \geq 2$ and $b \geq 1$. Let $\pi \in \mathcal{C} \cap S_n$ be a 3412 containing permutation (if there are no such permutations, then we are done by induction) and suppose that $\pi(i_1)\pi(i_2)\pi(i_3)\pi(i_4)$ is in the same relative order as 3412 where $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$. By induction we have the following (see Figure 7.3 for an illustration of these regions):

(i) $\pi([i_2] \times [\pi(i_4)])$ avoids $\ominus^{a+1}12$ and $\oplus^b 21$ so it can be covered by $f(a, b-1)$ monotonic rectangles,

(ii) $\pi([i_2, n] \times [\pi(i_1)])$ avoids $\ominus^a 12$ and $\oplus^{b+1}21$ so it can be covered by $f(a-1, b)$ monotonic rectangles,

(iii) $\pi([i_3] \times [\pi(i_4), n])$ avoids $\ominus^a 12$ and $\oplus^{b+1}21$ so it can be covered by $f(a-1, b)$ monotonic rectangles, and

(iv) $\pi([i_3, n] \times [\pi(i_1) \times n])$ avoids $\ominus^{a+1}12$ and $\oplus^b 21$ so it can be covered by $f(a, b-1)$ monotonic rectangles.

Because the four regions in (i)–(iv) cover π, it may be covered by $2f(a-1,b)+2f(a,b-1)$ monotonic rectangles. Furthermore, the 3412-avoiding permutations in $\mathcal{C}$ may be covered by $f(1,b) \leq f(a-1,b)$ monotonic rectangles by induction, so we may take $f(a,b) = 2f(a-1,b) + 2f(a,b-1)$, completing the proof. □

Before moving on let us note that the hypotheses of Theorem 7.2.3 can easily be checked by examining the basis of a finitely based class, and so griddability is decidable for such classes.

7.3 The Fibonacci dichotomy

The Fibonacci dichotomy for permutation classes, first proved by Kaiser and Klazar [84] states that all sub-Fibonacci permutation classes (those classes $\mathcal{C}$ for which $s_n(C)$ is less than the nth Fibonacci number for some n) have eventually polynomial enumeration.

Their proof uses results about Davenport-Schinzel sequences. Here we give a self-contained proof using the characterization of grid classes that breaks the result into its three constituent parts: sub-Fibonacci classes are griddable (Proposition 7.3.1), while sub-2^{n-1} griddable classes have a simple structure (Proposition 7.3.3), which implies that they have eventually polynomial enumeration (Theorem 7.3.4). We start with the first (and easiest) of these points.

Proposition 7.3.1. *All sub-Fibonacci classes are griddable.*

Proof. Let $\mathcal{C}$ be a sub-Fibonacci permutation class and suppose to the contrary that it is not griddable. Theorem 7.2.3 then implies that $\mathcal{C}$ contains $\ominus^a 12$ or $\oplus^a 21$ for all values of a. Thus we may assume by symmetry that $\mathcal{C}$ contains the class $\bigoplus\{1, 21\}$. But this class is enumerated by the Fibonacci numbers (this can be observed rather easily, or one may apply Proposition 1.5.2), and this is a contradiction. □

Our aim now is to characterize the griddable classes that lie in $\text{Grid}(M)$ for some matrix M for which $G(M)$ is a matching. For brevity, we will refer to such a grid class simply as the *grid class of a matching*.

We define a *vertical alternation* to be a permutation $\pi \in S_{2n}$ so that $\pi(1)$, $\pi(3)$, $\ldots$, $\pi(2n-1) < \pi(2)$, $\pi(4)$, $\ldots$, $\pi(2n)$. A *horizontal alternation* is then the inverse of a vertical alternation. Examples are shown in Figure 7.4. We begin by observing that classes with arbitrarily long alternations are not small.

Proposition 7.3.2. *If the permutation class $\mathcal{C}$ contains arbitrarily long alternations, then $s_n(\mathcal{C}) \geq 2^{n-1}$ for all n.*

Proof. By symmetry, let us suppose that $\mathcal{C}$ contains arbitrarily long horizontal alternations. By the Erdős-Szekeres theorem, $\mathcal{C}$ contains arbitrarily long horizontal alternations in which both sides are monotone. Therefore $\mathcal{C}$ contains an entire W class, either $W(1,1)$, $W(1,-1)$, $W(-1,1)$, or $W(-1,-1)$. It is easily to compute that the first and last of these classes have $2^n - n$ permutations of length n for $n \geq 1$ while the second and third have 2^{n-1} permutations of length n, establishing the proposition. □

This proposition shows that the sub-Fibonacci classes cannot contain arbitrarily long alternations.

Recalling the notion of intervals from Section 3.2, we say that a set of indices $\{i_1, i_2, \ldots, i_s\}$ in π is an *uninterrupted monotone interval* if $|i_{j+1} - i_j| = 1$ and $|\pi(i_{j+1}) - \pi(i_j)| = 1$ for all $j \in [s-1]$. Note that if $G(M)$ is a matching, then an M-gridding of π is a division of the elements of π into uninterrupted monotone intervals. Conversely, every division of π into uninterrupted monotone intervals gives an M-gridding of π for some M where $G(M)$ is a matching.

Proposition 7.3.3. *A griddable class lies in the grid class of a matching if and only if it does not contain arbitrarily long alternations.*

Proof. One direction is obvious: if a permutation class contains arbitrarily long alternations then it cannot lie in the grid class of a matching. We claim that the other direction is just as intuitive, but a formal proof takes a slight amount of effort.

Let $\mathcal{C} \subseteq \text{Grid}(N)$ for some $t \times u$ matrix N, and suppose that $\mathcal{C}$ does not contain any alternations (either horizontal or vertical) with more than d elements. It suffices to show that there is a constant m such that every permutation $\pi \in \mathcal{C}$ lies in $\text{Grid}(M)$ where $G(M)$ is a matching and M (which we allow to depend on π) has at most m nonzero entries. This is because we can ignore all 0 rows and columns, so the size of M can be bounded, and then there are only finitely many such matrices, so $\mathcal{C}$ will lie in the grid class of their direct sum (which also has a matching for its graph). Equivalently, by our remarks above, it suffices to show that every permutation in $\mathcal{C}$ can be divided into a bounded number of uninterrupted monotone intervals.

To this end, take some permutation $\pi \in \mathcal{C}$ of length n with N-gridding given by $1 = c_1 \leq c_2 \leq \cdots \leq c_{t+1} = n+1$ and $1 = r_1 \leq r_2 \leq \cdots \leq r_{u+1} = n+1$ and consider a particular block in this gridding, say

$$\pi^{(k,\ell)} := \pi([c_k, c_{k+1}) \times [r_\ell, r_{\ell+1})).$$

We will consider four types of alternations that elements of this block can participate in: vertical alternations either with blocks of the form $\pi^{(k,\ell^+)}$ for $\ell^+ > \ell$ or of the form

$\pi^{(k,\ell^-)}$ for $\ell^- < \ell$, and horizontal alternations with blocks of the form $\pi^{(k^+,\ell)}$ for $k^+ > k$ or of the form $\pi^{(k^-,\ell)}$ for $k^- < k$.

Every time that two consecutive elements in a block are separated either horizontally or vertically (that is, every time that two consecutive elements in a block fail to lie in an uninterrupted monotone interval together) then they contribute to the length of at least one of these four alternations. Therefore, at most $4d$ such separations can occur, so $\pi^{(k,\ell)}$ can be divided up into at most $4d+1$ uninterrupted monotone intervals. Hence π itself can be divided into at most $(4d+1)tu$ uninterrupted monotone intervals, proving the proposition. □

Having established that sub-2^{n-1} griddable classes (and in particular, sub-Fibonacci classes) lie in grid classes of matchings, it remains only to show that such grid classes (and their subclasses) have nice enumeration.

Theorem 7.3.4. *If the permutation class $\mathcal{C}$ lies in the grid class of a matching then there is a polynomial $p(n)$ so that $s_n(\mathcal{C}) = p(n)$ for all sufficiently large n.*

Proof. Let M be a $t \times t$ matrix whose graph is a matching (such matrices are necessarily square), let $\mathcal{C}$ be a subclass of Grid(M), and let $\pi \in \mathcal{C}$. We define the *greedy M-gridding* of π to be the gridding given by $1 = c_1 \leq c_2 \leq \cdots \leq c_{t+1} = n+1$ (the column divisions) and $1 = r_1 \leq r_2 \leq \cdots \leq r_{t+1} = n+1$ (the row divisions) where for each k, c_k is chosen so as to maximize $c_1 + c_2 + \cdots + c_k$. Because $G(M)$ is a matching, this uniquely defines the r's.

We now define a *peg point* of π to be a point which is either first or last (either horizontally or vertically; since the blocks are monotone, it doesn't matter) in its block in the greedy M-gridding of π. An example is shown in Figure 7.5. The *peg permutation*, ρ^π, of π is then the permutation formed by its peg points. We also associate to each permutation $\pi \in \mathcal{C}$ its *non-peg vector* $\mathbf{y}^\pi = (y_1, y_2, \ldots, y_t)$, where y_i denotes the number of non-peg points in $\pi([c_i, c_{i+1}) \times [n])$. Because the M-gridding is greedy, the pair $(\rho^\pi, \mathbf{y}^\pi)$ uniquely determines π.

We now partition the class $\mathcal{C}$ based upon peg permutations. Since there can be at most 3^t different peg permutations of members of $\mathcal{C}$ (for every column of M a peg permutation can have 0, 1, or 2 elements), this is a partition into a finite number of subsets. Let $\mathcal{C}^\rho$ denote the subset of $\mathcal{C}$ with peg permutation ρ. This is not a permutation class (the peg permutation of $\sigma \leq \pi$ need not be the peg permutation of π), but the set of non-peg vectors of permutations in this class, $\{\mathbf{y}^\pi : \pi \in \mathcal{C}^\rho\}$, is a closed class of vectors in $\mathbb{N}^t$. Therefore Theorem 1.6.1 shows that $\mathcal{C}^\rho$ has eventually polynomial enumeration, and so $\mathcal{C}$ does as well. □

As a consequence, we now have our proof of the Fibonacci dichotomy:

Corollary 7.3.5. *For every permutation class $\mathcal{C}$, one of the following occurs:*

- *$s_n(\mathcal{C}) \geq F_n$ for all n, or*
- *there is a polynomial $p(n)$ so that $s_n(\mathcal{C}) = p(n)$ for all sufficiently large n.*

References

[1] Albert, M., Elder, M., Rechnitzer, A., Westcott, P., and Zabrocki, M. On the Wilf-Stanley limit of 4231-avoiding permutations and a conjecture of arratia. arXiv:math.CO/0502504.

[2] Albert, M. H., Aldred, R. E. L., Atkinson, M. D., van Ditmarsch, H. P., Handley, C. C., and Holton, D. A. Restricted permutations and queue jumping. *Discrete Math. 287*, 1-3 (2004), 129–133.

[3] Albert, M. H., and Atkinson, M. D. Simple permutations and pattern restricted permutations. *Discrete Math. 300*, 1-3 (2005), 1–15.

[4] Albert, M. H., Atkinson, M. D., Handley, C. C., Holton, D. A., and Stromquist, W. On packing densities of permutations. *Electron. J. Combin. 9*, 1 (2002), Research Paper 5, 20 pp. (electronic).

[5] Albert, M. H., Atkinson, M. D., and Klazar, M. The enumeration of simple permutations. *J. Integer Seq. 6*, 4 (2003), Article 03.4.4, 18 pp. (electronic).

[6] Albert, M. H., Atkinson, M. D., and Ruškuc, N. Regular closed sets of permutations. *Theoret. Comput. Sci. 306*, 1-3 (2003), 85–100.

[7] Albert, M. H., Linton, S., and Ruškuc, N. The insertion encoding of permutations. Preprint.

[8] Alon, N., and Friedgut, E. On the number of permutations avoiding a given pattern. *J. Combin. Theory Ser. A 89*, 1 (2000), 133–140.

[9] Alon, N., and Spencer, J. H. *The probabilistic method*, second ed. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience [John Wiley & Sons], New York, 2000.

[10] Arratia, R. On the Stanley-Wilf conjecture for the number of permutations avoiding a given pattern. *Electron. J. Combin. 6* (1999), Note, N1, 4 pp. (electronic).

[11] Atkinson, M. D. Permutations which are the union of an increasing and a decreasing subsequence. *Electron. J. Combin. 5* (1998), Research paper 6, 13 pp. (electronic).

[12] Atkinson, M. D. Restricted permutations. *Discrete Math. 195*, 1-3 (1999), 27–38.

[13] Atkinson, M. D., Murphy, M. M., and Ruškuc, N. Partially well-ordered closed sets of permutations. *Order 19*, 2 (2002), 101–113.

[14] Atkinson, M. D., Murphy, M. M., and Ruškuc, N. Sorting with two ordered stacks in series. *Theoret. Comput. Sci. 289*, 1 (2002), 205–223.

[15] Atkinson, M. D., Murphy, M. M., and Ruškuc, N. Pattern avoidance classes and subpermutations. *Electron. J. Combin. 12*, 1 (2005), Research paper 60, 18 pp. (electronic).

[16] Atkinson, M. D., and Stitt, T. Restricted permutations and the wreath product. *Discrete Math. 259*, 1-3 (2002), 19–36.

[17] Babson, E., and West, J. The permutations $123p_4 \cdots p_m$ and $321p_4 \cdots p_m$ are Wilf-equivalent. *Graphs Combin. 16*, 4 (2000), 373–380.

[18] Backelin, J., West, J., and Xin, G. Wilf-equivalence for singleton classes. Preprint.

[19] Baik, J., Deift, P., and Johansson, K. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc. 12*, 4 (1999), 1119–1178.

[20] Banderier, C., Bousquet-Mélou, M., Denise, A., Flajolet, P., Gardy, D., and Gouyou-Beauchamps, D. Generating functions for generating trees. *Discrete Math. 246*, 1-3 (2002), 29–55.

[21] Barcucci, E., Del Lungo, A., Pergola, E., and Pinzani, R. ECO: a methodology for the enumeration of combinatorial objects. *J. Differ. Equations Appl. 5*, 4-5 (1999), 435–490.

[22] Barcucci, E., Del Lungo, A., Pergola, E., and Pinzani, R. From Motzkin to Catalan permutations. *Discrete Math. 217*, 1-3 (2000), 33–49.

[23] Barcucci, E., Del Lungo, A., Pergola, E., and Pinzani, R. Permutations avoiding an increasing number of length-increasing forbidden subsequences. *Discrete Math. Theor. Comput. Sci. 4*, 1 (2000), 31–44 (electronic).

[24] Barcucci, E., Pergola, E., Pinzani, R., and Rinaldi, S. ECO method and hill-free generalized Motzkin paths. *Sém. Lothar. Combin. 46* (2001/02), Art. B 46b, 14 pp. (electronic).

[25] Bergeron, F., Bousquet-Mélou, M., and Dulucq, S. Standard paths in the composition poset. *Ann. Sci. Math. Québec 19*, 2 (1995), 139–151.

[26] Billey, S. C., and Warrington, G. S. Kazhdan-Lusztig polynomials for 321-hexagon-avoiding permutations. *J. Algebraic Combin. 13*, 2 (2001), 111–136.

[27] Björner, A., and Sagan, B. Rationality of the Möbius function of a composition poset. arXiv:math.CO/0510282.

[28] Blackwell, P. An alternative proof of a theorem of Erdős and Szekeres. *Amer. Math. Monthly 78* (1971), 273.

[29] Bóna, M. New records in Stanley-Wilf limits. Preprint.

[30] BÓNA, M. Exact enumeration of 1342-avoiding permutations: a close link with labeled trees and planar maps. *J. Combin. Theory Ser. A 80*, 2 (1997), 257–272.

[31] BÓNA, M. The number of permutations with exactly r 132-subsequences is P-recursive in the size! *Adv. in Appl. Math. 18*, 4 (1997), 510–522.

[32] BÓNA, M. The solution of a conjecture of Stanley and Wilf for all layered patterns. *J. Combin. Theory Ser. A 85*, 1 (1999), 96–104.

[33] BÓNA, M. A survey of stack-sorting disciplines. *Electron. J. Combin. 9*, 2 (2002/03), Article 1, 16 pp. (electronic).

[34] BÓNA, M. *Combinatorics of permutations*. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2004. With a foreword by Richard Stanley.

[35] BÓNA, M. The limit of a Stanley-Wilf sequence is not always rational, and layered patterns beat monotone patterns. *J. Combin. Theory Ser. A 110*, 2 (2005), 223–235.

[36] BONIZZONI, P., AND MCCONNELL, R. M. Nesting of prime substructures in k-ary relations. *Theoret. Comput. Sci. 259*, 1-2 (2001), 341–357.

[37] BOSE, P., BUSS, J. F., AND LUBIW, A. Pattern matching for permutations. *Inform. Process. Lett. 65*, 5 (1998), 277–283.

[38] BOUSQUET-MÉLOU, M. Four classes of pattern-avoiding permutations under one roof: generating trees with two labels. *Electron. J. Combin. 9*, 2 (2002/03), Research paper 19, 31 pp. (electronic).

[39] BOUSQUET-MÉLOU, M., AND STEINGRÍMSSON, E. Decreasing subsequences in permutations and Wilf-equivalence for involutions. arXiv:math.CO/0405334.

[40] BRLEK, S., DUCHI, E., PERGOLA, E., AND RINALDI, S. On the equivalence problem for succession rules. *Discrete Math. 298*, 1-3 (2005), 142–154.

[41] CAMERON, P. J. Homogeneous permutations. *Electron. J. Combin. 9*, 2 (2002/03), Research paper 2, 9 pp. (electronic).

[42] CHASE, P. J. Subsequence numbers and logarithmic concavity. *Discrete Math. 16*, 2 (1976), 123–140.

[43] CHEN, W. Y. C., MANSOUR, T., AND YAN, S. H. F. Matchings avoiding partial patterns. arXiv:math.CO/0504342.

[44] CHOW, T., AND WEST, J. Forbidden subsequences and Chebyshev polynomials. *Discrete Math. 204*, 1-3 (1999), 119–128.

[45] CHUDNOVSKY, M., CORNUÉJOLS, G., LIU, X., SEYMOUR, P. D., AND VUŠKOVIĆ, K. Recognizing Berge graphs. *Combinatorica 25*, 2 (2005), 143–186.

[46] Chudnovsky, M., Robertson, N., Seymour, P., and Thomas, R. The strong perfect graph theorem. Preprint, available at http://www.math.gatech.edu/ thomas/spgc.html.

[47] Chudnovsky, M., Robertson, N., Seymour, P. D., and Thomas, R. Progress on perfect graphs. *Math. Program. 97*, 1-2, Ser. B (2003), 405–422. ISMP, 2003 (Copenhagen).

[48] Chung, F. R. K., Graham, R. L., Hoggatt, Jr., V. E., and Kleiman, M. The number of Baxter permutations. *J. Combin. Theory Ser. A 24*, 3 (1978), 382–394.

[49] Corneil, D. G., Lerchs, H., and Burlingham, L. S. Complement reducible graphs. *Discrete Appl. Math. 3*, 3 (1981), 163–174.

[50] Corteel, S., Louchard, G., and Pemantle, R. Common intervals of permutations. In *Mathematics and computer science. III*, Trends Math. Birkhäuser, Basel, 2004, pp. 3–14.

[51] Deutsch, E., Hildebrand, A. J., and Wilf, H. S. Longest increasing subsequences in pattern-restricted permutations. *Electron. J. Combin. 9*, 2 (2002/03), Research paper 12, 8 pp. (electronic).

[52] Drucker, S., Fodor, I., and Fisher, E. On random walks and levels of the form n^α. *Rocky Mountain J. Math. 26*, 1 (1996), 117–134.

[53] Duchi, E., Fedou, J.-M., and Rinaldi, S. From object grammars to ECO systems. *Theoret. Comput. Sci. 314*, 1-2 (2004), 57–95.

[54] Dulucq, S., Gire, S., and Guibert, O. A combinatorial proof of J. West's conjecture. *Discrete Math. 187*, 1-3 (1998), 71–96.

[55] Dulucq, S., Gire, S., and West, J. Permutations with forbidden subsequences and nonseparable planar maps. *Discrete Math. 153*, 1-3 (1996), 85–103.

[56] Ehrenfeucht, A., Harju, T., and Rozenberg, G. *The theory of 2-structures*. World Scientific Publishing Co. Inc., River Edge, NJ, 1999.

[57] Ehrenfeucht, A., Harju, T., ten Pas, P., and Rozenberg, G. Permutations, parenthesis words, and Schröder numbers. *Discrete Math. 190*, 1-3 (1998), 259–264.

[58] Ehrenfeucht, A., and McConnell, R. A k-structure generalization of the theory of 2-structures. *Theoret. Comput. Sci. 132*, 1-2 (1994), 209–227.

[59] Ehrenfeucht, A., and Rozenberg, G. Primitivity is hereditary for 2-structures. *Theoret. Comput. Sci. 70*, 3 (1990), 343–358.

[60] Elder, M., and Vatter, V. Problems and conjectures presented at the Third International Conference on Permutation Patterns, University of Florida, March 7-11, 2005. arXiv:math.CO/0505504.

[61] Erdős, P., and Szekeres, G. A combinatorial problem in geometry. *Compos. Math. 2* (1935), 463–470.

[62] Erdős, P., Fried, E., Hajnal, A., and Milner, E. C. Some remarks on simple tournaments. *Algebra Universalis 2* (1972), 238–245.

[63] Ferrari, L., Pergola, E., Pinzani, R., and Rinaldi, S. An algebraic characterization of the set of succession rules. *Theoret. Comput. Sci. 281*, 1-2 (2002), 351–367.

[64] Ferrari, L., Pinzani, R., and Rinaldi, S. Enumerative results on integer partitions using the ECO method. In *Mathematics and computer science. III*, Trends Math. Birkhäuser, Basel, 2004, pp. 25–36.

[65] Földes, S. On intervals in relational structures. *Z. Math. Logik Grundlag. Math. 26*, 2 (1980), 97–101.

[66] Földes, S., and Hammer, P. L. Split graphs. In *Proceedings of the Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, La., 1977)* (Winnipeg, Man., 1977), Utilitas Math., pp. 311–315. Congressus Numerantium, No. XIX.

[67] Földes, S., and Hammer, P. L. The Dilworth number of a graph. *Ann. Discrete Math. 2* (1978), 211–219.

[68] Fraïssé, R. Sur l'extension aux relations de quelques propriétés des ordres. *Ann. Sci. Ecole Norm. Sup. (3) 71* (1954), 363–388.

[69] Fraïssé, R. *Cours de logique mathématique. Tome 1: Relation et formule logique.* Gauthier-Villars Éditeur, Paris, 1971. Deuxième édition revue et modifiée, Collection de "Logique Mathématique", Série A, No. 23.

[70] Fraïssé, R. Present problems about intervals in relation-theory and logic. In *Non-classical logics, model theory and computability (Proc. Third Latin-Amer. Sympos. Math. Logic, Campinas, 1976).* North-Holland, Amsterdam, 1977, pp. 179–200. Stud. Logic Found. Math., Vol. 89.

[71] Fraïssé, R. L'intervalle en théorie des relations; ses généralisations; filtre intervallaire et clôture d'une relation. In *Orders: description and roles (L'Arbresle, 1982)*, vol. 99 of *North-Holland Math. Stud.* North-Holland, Amsterdam, 1984, pp. 313–341.

[72] Fulmek, M. Enumeration of permutations containing a prescribed number of occurrences of a pattern of length three. *Adv. in Appl. Math. 30*, 4 (2003), 607–632.

[73] Füredi, Z., and Hajnal, P. Davenport-Schinzel theory of matrices. *Discrete Math. 103*, 3 (1992), 233–251.

[74] Gessel, I. M. Symmetric functions and P-recursiveness. *J. Combin. Theory Ser. A 53*, 2 (1990), 257–285.

[75] Goulden, I. P., and West, J. Raney paths and a combinatorial relationship between rooted nonseparable planar maps and two-stack-sortable permutations. *J. Combin. Theory Ser. A 75*, 2 (1996), 220–242.

[76] Green, R. M., and Losonczy, J. Freely braided elements of Coxeter groups. *Ann. Comb. 6*, 3-4 (2002), 337–348.

[77] Guibert, O., and Pergola, E. Enumeration of vexillary involutions which are equal to their mirror/complement. *Discrete Math. 224*, 1-3 (2000), 281–287.

[78] Guruswami, V. Enumerative aspects of certain subclasses of perfect graphs. *Discrete Math. 205*, 1-3 (1999), 97–117.

[79] Hammersley, J. M. A few seedlings of research. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. I: Theory of statistics* (Berkeley, Calif., 1972), Univ. California Press, pp. 345–394.

[80] Higman, G. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc. (3) 2* (1952), 326–336.

[81] Hodges, W. *Model theory*, vol. 42 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[82] Huczynska, S., and Ruškuc, N. Pattern classes of permutations via bijections between linearly ordered sets. Preprint.

[83] Jani, M., and Rieper, R. G. Continued fractions and Catalan problems. *Electron. J. Combin. 7* (2000), Research paper 45, 8 pp. (electronic).

[84] Kaiser, T., and Klazar, M. On growth rates of closed permutation classes. *Electron. J. Combin. 9*, 2 (2002/03), Research paper 10, 20 pp. (electronic).

[85] Kaplansky, I. The asymptotic distribution of runs of consecutive elements. *Ann. Math. Statistics 16* (1945), 200–203.

[86] Kézdy, A. E., Snevily, H. S., and Wang, C. Partitioning permutations into increasing and decreasing subsequences. *J. Combin. Theory Ser. A 73*, 2 (1996), 353–359.

[87] Klazar, M. The Füredi-Hajnal conjecture implies the Stanley-Wilf conjecture. In *Formal power series and algebraic combinatorics (Moscow, 2000)*. Springer, Berlin, 2000, pp. 250–255.

[88] Klazar, M. On the least exponential growth admitting uncountably many closed permutation classes. *Theoret. Comput. Sci. 321*, 2-3 (2004), 271–281.

[89] Knuth, D. E. *The art of computer programming. Vol. 1: Fundamental algorithms.* Second printing. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont, 1969.

[90] Krattenthaler, C. Permutations with restricted patterns and Dyck paths. *Adv. in Appl. Math. 27*, 2-3 (2001), 510–530.

[91] Kremer, D. Permutations with forbidden subsequences and a generalized Schröder number. *Discrete Math. 218*, 1-3 (2000), 121–130.

[92] Kremer, D., and Shiu, W. C. Finite transition matrices for permutations avoiding pairs of length four patterns. *Discrete Math. 268*, 1-3 (2003), 171–183.

[93] Kruskal, J. B. The theory of well-quasi-ordering: A frequently discovered concept. *J. Combinatorial Theory Ser. A 13* (1972), 297–305.

[94] Le, I. Wilf classes of pairs of permutations of length 4. *Electron. J. Combin. 12* (2005), Research Paper 25, 27 pp. (electronic).

[95] Lipson, M. Completion of the wilf-classification of 3-5 pairs using generating trees. Preprint.

[96] Lothaire, M. *Combinatorics on words*, vol. 17 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1983.

[97] Lovász, L. A note on the line reconstruction problem. *J. Combinatorial Theory Ser. B 13* (1972), 309–310.

[98] MacMahon, P. A. *Combinatory Analysis*. Cambridge University Press, London, 1915/16.

[99] Mansour, T. Permutations containing and avoiding certain patterns. In *Formal power series and algebraic combinatorics (Moscow, 2000)*. Springer, Berlin, 2000, pp. 704–708.

[100] Mansour, T. On an open problem of Green and Losonczy: exact enumeration of freely braided permutations. *Discrete Math. Theor. Comput. Sci. 6*, 2 (2004), 461–470.

[101] Mansour, T., and Stankova, Z. 321-polygon-avoiding permutations and Chebyshev polynomials. *Electron. J. Combin. 9*, 2 (2002/03), Research paper 5, 16 pp. (electronic).

[102] Mansour, T., and Vainshtein, A. Restricted permutations, continued fractions, and Chebyshev polynomials. *Electron. J. Combin. 7* (2000), Research Paper 17, 9 pp. (electronic).

[103] Mansour, T., and Vainshtein, A. Restricted permutations and Chebyshev polynomials. *Sém. Lothar. Combin. 47* (2001/02), Article B47c, 17 pp. (electronic).

[104] Mansour, T., and Vainshtein, A. Counting occurrences of 132 in a permutation. *Adv. in Appl. Math. 28*, 2 (2002), 185–195.

[105] Marcus, A., and Tardos, G. Excluded permutation matrices and the Stanley-Wilf conjecture. *J. Combin. Theory Ser. A 107*, 1 (2004), 153–160.

[106] Marinov, D., and Radoičić, R. Counting 1324-avoiding permutations. *Electron. J. Combin. 9*, 2 (2002/03), Research paper 13, 9 pp. (electronic).

[107] Merlini, D., Sprugnoli, R., and Verri, M. C. An algebra for proper generating trees. In *Mathematics and computer science (Versailles, 2000)*, Trends Math. Birkhäuser, Basel, 2000, pp. 127–139.

[108] Merlini, D., Sprugnoli, R., and Verri, M. C. The tennis ball problem. *J. Combin. Theory Ser. A 99*, 2 (2002), 307–344.

[109] Möhring, R. H. On the distribution of locally undecomposable relations and independence systems. In *Colloquium on Mathematical Methods of Operations Research (Aachen, 1980)*, vol. 42 of *Methods Oper. Res.* Athenäum/Hain/Hanstein, Königstein/Ts., 1981, pp. 33–48.

[110] Möhring, R. H. Algorithmic aspects of the substitution decomposition in optimization over relations, sets systems and Boolean functions. *Ann. Oper. Res. 4*, 1-4 (1985), 195–225.

[111] Möhring, R. H., and Radermacher, F. J. Substitution decomposition for discrete structures and connections with combinatorial optimization. In *Algebraic and combinatorial methods in operations research*, vol. 95 of *North-Holland Math. Stud.* North-Holland, Amsterdam, 1984, pp. 257–355.

[112] Murphy, M. M. *Restricted permutations, antichains, atomic classes, and stack sorting*. PhD thesis, Univ. of St. Andrews, 2002.

[113] Murphy, M. M., and Vatter, V. Profile classes and partial well-order for permutations. *Electron. J. Combin. 9*, 2 (2002/03), Research paper 17, 30 pp. (electronic).

[114] Nash-Williams, C. S. J. A. On well-quasi-ordering finite trees. *Proc. Cambridge Philos. Soc. 59* (1963), 833–835.

[115] Noonan, J. The number of permutations containing exactly one increasing subsequence of length three. *Discrete Math. 152*, 1-3 (1996), 307–313.

[116] Noonan, J., and Zeilberger, D. The enumeration of permutations with a prescribed number of "forbidden" patterns. *Adv. in Appl. Math. 17*, 4 (1996), 381–407.

[117] O'Hara, K. M. Unimodality of Gaussian coefficients: a constructive proof. *J. Combin. Theory Ser. A 53*, 1 (1990), 29–52.

[118] Pergola, E., and Sulanke, R. A. Schröder triangles, paths, and parallelogram polyominoes. *J. Integer Seq. 1* (1998), Article 98.1.7 (9 HTML documents).

[119] Pouzet, M., and Rosenberg, I. G. Sperner properties for groups and relations. *European J. Combin. 7*, 4 (1986), 349–370.

[120] Pratt, V. R. Computing permutations with double-ended queues, parallel stacks and parallel queues. In *STOC '73: Proceedings of the fifth annual ACM symposium on Theory of computing* (New York, NY, USA, 1973), ACM Press, pp. 268–277.

[121] Proctor, R. A. Solution of two difficult combinatorial problems with linear algebra. *Amer. Math. Monthly 89*, 10 (1982), 721–734.

[122] Rao, S. B. Towards a theory of forcibly hereditary P-graphic sequences. In *Combinatorics and graph theory (Calcutta, 1980)*, vol. 885 of *Lecture Notes in Math.* Springer, Berlin, 1981, pp. 441–458.

[123] Ray, N., and West, J. Posets of matrices and permutations with forbidden subsequences. *Ann. Comb. 7*, 1 (2003), 55–88.

[124] Regev, A. Asymptotic values for degrees associated with strips of Young diagrams. *Adv. in Math. 41*, 2 (1981), 115–136.

[125] Robertson, A., Wilf, H. S., and Zeilberger, D. Permutation patterns and continued fractions. *Electron. J. Combin. 6* (1999), Research Paper 38, 6 pp. (electronic).

[126] Sagan, B. E., and Vatter, V. The möbius function of the composition poset. *J. Alg. Comb.*, submitted.

[127] Savage, C. D., and Wilf, H. S. Pattern avoidance in compositions and multiset permutations. arXiv:math.CO/0504310.

[128] Scheinerman, E. R. On the structure of hereditary classes of graphs. *J. Graph Theory 10*, 4 (1986), 545–551.

[129] Schmerl, J. H., and Trotter, W. T. Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures. *Discrete Math. 113*, 1-3 (1993), 191–205.

[130] Simion, R., and Schmidt, F. W. Restricted permutations. *European J. Combin. 6*, 4 (1985), 383–406.

[131] Sloane, N. J. A. The on-line encyclopedia of integer sequences. Available online at http://www.research.att.com/ñjas/sequences/, 2005.

[132] Snellman, J. Saturated chains in composition posets. `arXiv:math.CO/0505262`.

[133] Snellman, J. Standard paths in another composition poset. *Electron. J. Combin. 11*, 1 (2004), Research Paper 76, 8 pp. (electronic).

[134] Spielman, D. A., and Bóna, M. An infinite antichain of permutations. *Electron. J. Combin. 7* (2000), Note 2, 4 pp. (electronic).

[135] Stankova, Z. Classification of forbidden subsequences of length 4. *European J. Combin. 17*, 5 (1996), 501–517.

[136] Stankova, Z., and West, J. A new class of Wilf-equivalent permutations. *J. Algebraic Combin. 15*, 3 (2002), 271–290.

[137] Stankova, Z., and West, J. Explicit enumeration of 321, hexagon-avoiding permutations. *Discrete Math. 280*, 1-3 (2004), 165–189.

[138] Stankova, Z. E. Forbidden subsequences. *Discrete Math. 132*, 1-3 (1994), 291–316.

[139] Stanley, R. P. Enumeration of posets generated by disjoint unions and ordinal sums. *Proc. Amer. Math. Soc. 45* (1974), 295–299.

[140] Stanley, R. P. Solution to problem E2546. *Amer. Math. Monthly 83*, 10 (1976), 813–814.

[141] Stanley, R. P. Weyl groups, the hard Lefschetz theorem, and the Sperner property. *SIAM J. Algebraic Discrete Methods 1*, 2 (1980), 168–184.

[142] Stanley, R. P. Quotients of Peck posets. *Order 1*, 1 (1984), 29–34.

[143] Stanley, R. P. Log-concave and unimodal sequences in algebra, combinatorics, and geometry. In *Graph theory and its applications: East and West (Jinan, 1986)*, vol. 576 of *Ann. New York Acad. Sci.* New York Acad. Sci., New York, 1989, pp. 500–535.

[144] Stanley, R. P. *Enumerative combinatorics. Vol. 1*, vol. 49 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1997.

[145] Stanley, R. P. *Enumerative combinatorics. Vol. 2*, vol. 62 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1999.

[146] Stanley, R. P. Recent progress in algebraic combinatorics. *Bull. Amer. Math. Soc. (N.S.) 40*, 1 (2003), 55–68 (electronic).

[147] Stanton, D. Unimodality and Young's lattice. *J. Combin. Theory Ser. A 54*, 1 (1990), 41–53.

[148] Sylvester, J. J. Proof of the hitherto undemonstrated fundamental theorem of invariants. In *The collected mathematical papers of James Joseph Sylvester*, vol. 3. Cambridge University Press, Chelsea, New York, 1973, pp. 117–126.

[149] Tardos, G. On 0-1 matrices and small excluded submatrices. *J. Combin. Theory Ser. A 111*, 2 (2005), 266–288.

[150] Tarjan, R. E. Sorting using networks of queues and stacks. *J. Assoc. Comput. Mach. 19* (1972), 341–346.

[151] Tenner, B. E. Reduced decompositions and permutation patterns. arXiv:math.CO/0506242.

[152] Uno, T., and Yagiura, M. Fast algorithms to enumerate all common intervals of two permutations. *Algorithmica 26*, 2 (2000), 290–309.

[153] Usha Devi, N., and Vijayakumar, G. R. Hereditary properties of graphs. *Discrete Math. 206*, 1-3 (1999), 213–215.

[154] Vatter, V. Enumeration schemes for restricted permutations. arXiv:math.CO/0510044.

[155] Vatter, V. Finitely labelled generating trees and restricted permutations. *J. Symb. Comp.*, to appear.

[156] Vatter, V. Permutations avoiding two patterns of length three. *Electron. J. Combin. 9*, 2 (2002/03), Research paper 6, 19 pp. (electronic).

[157] West, J. *Permutations with forbidden subsequences and stack-sortable permutations*. PhD thesis, M.I.T., 1990.

[158] West, J. Generating trees and the Catalan and Schröder numbers. *Discrete Math. 146*, 1-3 (1995), 247–262.

[159] West, J. Generating trees and forbidden subsequences. *Discrete Math. 157*, 1-3 (1996), 363–374.

[160] Wilf, H. S. What is an answer? *Amer. Math. Monthly 89*, 5 (1982), 289–292.

[161] Wilf, H. S. The patterns of permutations. *Discrete Math. 257*, 2-3 (2002), 575–583.

[162] Wolfowitz, J. Note on runs of consecutive elements. *Ann. Math. Statistics 15* (1944), 97–98.

[163] Zeilberger, D. A loving rendition of the Marcus-Tardos amazing proof of the Füredi-Hajnal conjecture. Available at http://www.math.rutgers.edu/ zeilberg/mamarim/mamarimhtml/martar.html.

[164] Zeilberger, D. A snappy proof that 123-avoiding words are equinumerous with 132-avoiding words. Available at http://www.math.rutgers.edu/ zeilberg/mamarim/mamarimhtml/a123.html.

[165] Zeilberger, D. A proof of Julian West's conjecture that the number of two-stack-sortable permutations of length n is $2(3n)!/((n+1)!(2n+1)!)$. *Discrete Math. 102*, 1 (1992), 85–93.

[166] Zeilberger, D. Enumeration schemes and, more importantly, their automatic generation. *Ann. Comb. 2*, 2 (1998), 185–195.