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ABSTRACT OF THE DISSERTATION

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This thesis concerns two problems at the intersection of discrete mathematics and theoretical computer science. Broadly speaking, both of these problems regard the outcome of a random process on a discrete structure.

It is well known that any graph of maximum degree Δ can be properly edge colored with at most $\Delta + 1$ colors. In the online setting, it has been a matter of some interest to find an algorithm that can properly edge color any graph on n vertices with maximum degree $\Delta = \omega(\log n)$ using at most $(1 + o(1))\Delta$ colors. Here we study the naïve random greedy algorithm, which simply chooses a legal color uniformly at random for each edge upon arrival. We show that this algorithm can $(1 + \epsilon)\Delta$-color a graph for arbitrary ϵ in two contexts: first, if the graph is fixed ahead of time and its edges arrive in a uniformly random order, and second, if the edges of the graph are selected adaptively by an adversary as the algorithm progresses, but with the restriction that $n = O(\Delta)$. A corollary of the second result is that there must exist a deterministic algorithm to $(1 + \epsilon)\Delta$-color dense graphs.

For the second problem, we study probability distributions on a set A with an associated weight function $w : A \to [0, 1]$. For any such distribution, μ, we define a defending distribution to be a distribution on subsets of A of total weight at most 1 that satisfies certain constraints with respect to both μ and linear orderings on A.

The existence of a defending distribution is relevant to the field of social choice theory, where it was shown in a paper by Jiang, Munagala, and Wang that it implies the existence of a stable lottery, a method of selecting committees that is "fair" in some sense. We will extend a result from that paper to show that under certain restrictions of the weight function, a defending distribution always exists.

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Chapter 1

Introduction

This thesis is divided primarily into two sections, each of which discusses a problem in the area of discrete math and theoretical computer science. The first studies a simple online algorithm for graph edge coloring. The second studies probability distributions of sets of weighted elements and arises from a problem in computational social choice theory.

1.1 Greedy Online Edge Coloring

The first problem we consider is related to edge coloring of graphs. A simple argument shows that any graph with maximum degree Δ can be deterministically edge colored with at most $2\Delta - 1$ colors by the simple greedy algorithm. In fact, Vizing's famous theorem tells us that any graph with maximum degree Δ has chromatic index exactly Δ or $\Delta+1$, and his proof shows that any such graph can be $(\Delta + 1)$-edge colored deterministically in polynomial time. In contrast, in the online setting Bar-Noy, Motwani, and Naor showed that no randomized online algorithm using fewer than $2\Delta-1$ colors to color graphs of maximum degree Δ on n vertices when $\Delta = O(\sqrt{\log n})$ can succeed with probability more than $1 - \frac{1}{e}$. However, in the case that $\Delta = \omega(\log n)$, they conjectured that the random greedy algorithm could successfully $(1+o(1))\Delta$-color graphs of maximum degree Δ on n vertices. In the second chapter of this thesis, we study the random greedy algorithm using $(1+\epsilon)\Delta$ colors (for ϵ an arbitrarily small constant).

There are many reasons this algorithm in particular is inherently worth studying, starting with its simplicity. More than that, however, the outcome of this algorithm is interesting in that each step is very dependent on previous steps. Previous work utilizing random colorings has often relied on independence or near-independence of consecutive choices, so the analysis of this algorithm required the development of a different method of proof. We prove guarantees on the performance of this algorithm in two different settings. First, we consider the case where the edges arrive in a uniformly random order. In this case, we obtain the following result:

Theorem 1 (Informal version of Theorem 6)**.** *(***Random order case***) Let $\epsilon \in (0,1)$ be a constant. The random greedy algorithm using $(1+\epsilon)\Delta$ colors, when given any simple graph G of maximum degree $\Delta = \omega(\log n)$ whose edges are presented in a uniformly random order, successfully edge colors G with high probability.*

In the second setting we allow the edges to be chosen by an *adaptive adversary*. This means that the graph G is not fixed ahead of time, and the choice of each successive edge can depend on the colors assigned to previous edges. In this case, we are also able to show that the algorithm is likely to succeed, but we require that the resulting graph G be dense.

Theorem 2 (Informal version of Theorem 7)**.** *(***Dense case***) Let $\epsilon \in (0,1)$, $M \geq 1$ be constants. Suppose G is an* adaptively chosen *simple graph with maximum degree Δ and $n \leq M\Delta$. Then, the random greedy algorithm using $(1+\epsilon)\Delta$ colors succeeds in edge coloring G with high probability.*

Allowing the graph G to be adaptively chosen is a very powerful assumption; the work of Ben-David et. al. in [13] shows that this removes the power of randomization altogether. Thus, we obtain the following corollary to Theorem 2:

Corollary 1. Let $\epsilon \in (0,1)$ be a constant. For Δ sufficiently large and $n = O(\Delta)$ there is a deterministic online algorithm that given any simple graph with n vertices and maximum degree Δ will produce a valid coloring using $(1+\epsilon)\Delta$ colors.

As mentioned above, the analysis of the random greedy algorithm is difficult due to the inherent dependencies between choices at each step. Nevertheless, we manage to show that for each vertex, the set of colors assigned by this algorithm to its adjacent edges appears random in some sense. More precisely, we devise a measure for the distance between the coloring produced by this algorithm at a vertex and a coupled, uniformly random coloring (independent from the coloring on the rest of the graph), at each point in time. Roughly, this measure can be decomposed into two main parts: a part that behaves nicely, similar to an independent process, and a part that depends on the neighbors of the vertex. We bound the first part (for most vertices) using a novel martingale concentration lemma, and then use an inductive argument to bound the second part and thus obtain an upper bound in this measure for all vertices. We are then able to use this similarity to an independent random coloring to deduce that the algorithm succeeds with high probability at each vertex, and therefore on the entire graph.

1.2 Existence of Defending Distributions

The second problem discussed in this thesis also studies the outcome of a random process on a discrete structure. Given a finite set A, associated weight function $w : A \to (0,1)$, and probability distribution μ on subsets of A, we define a *defending distribution* ν to be a distribution on supported on subsets of total weight at most 1 that satisfies the following condition: for any total ordering $\succ$ on A,

$$\Pr_{a\sim\mu, S\sim\nu}[a \succ a' \;\forall\; a' \in S] < \mathop{\mathbb{E}}_{a\sim\mu} w(a). \tag{1.1}$$

It is not difficult to see that if all elements of A have the same weight, then for any distribution μ on subsets of A, there must exist a defending distribution ν (see Lemma 39.) The following conjecture is (rephrased) from [45].

Conjecture 1. For any weight function w and any distribution μ on A, there exists a defending distribution ν for μ.

In the third chapter, we prove this conjecture in the following two restricted settings:

Theorem 3. *Suppose w is supported on any two values $\{k_1, k_2\}$. Then, for any distribution μ on A, there exists a defending distribution ν for μ.*

Theorem 4. *Suppose w is supported on any three values $\{\frac{1}{b_1}, \frac{1}{b_1 b_2}, \frac{1}{b_1 b_2 b_3}\}$, where $b_1, b_2, b_3 \in \mathbb{Z}_{\geq 2}$. Then, for any distribution μ on A, there exists a defending distribution ν for μ.*

We are also able to extend the covered cases to include the following, more specific, setting:

Theorem 5. *Suppose w is supported on three weights $k_1 > k_2 > k_3$ s.t. $k_1 \in (\frac{1}{2}, 1)$, $k_2 \in (\frac{1}{3}, \frac{1}{2}]$, and $k_1 + k_2 > 1$. Then for any distribution μ on A, it is possible to find a defending distribution.*

Our main method of proof is as follows: given a distribution μ, we choose a distribution ν that selects multiple elements from μ independently at random. Then, for a weight function supported on ℓ distinct values, we look at the distributions μ_i of elements drawn from μ conditioned on being of the i^{th} highest weight. Given an ordering $\succ$, we consider quantities of the form $f_i(x_1, x_2, ..., x_\ell)$, the probability that if we draw x_j elements from each μ_j, all independently, that the first element drawn from μ_i is the greatest, according to $\succ$. It is not hard to show that for a distribution ν of the form above, the left hand side of Equation (1.1) can be written in terms of such quantities. We then use a coupling argument to derive a system of upper bounds on these terms. As a result, the problem of finding a defending distribution becomes one of solving a system of linear equations to combine these upper bounds and obtain Equation (1.1). In the case of Theorem 4, we use the Max Flow Min Cut Theorem from graph theory to prove the existence of such a solution.

As shown in [45], the existence of a defending distribution ν for all sets A, weight functions w, and distributions μ has implications in the field of social choice theory, where it proves that in the committee selection problem, a stable lottery is always guaranteed to exist.

Chapter 2

Greedy Online Edge Coloring

This chapter is joint work with Aditi Dudeja and Michael Saks ([33]).

2.1 Introduction

The edge coloring problems for graphs is to assign colors to the edges of a given graph so that any two edges meeting at a vertex are assigned different colors. Trivially, the number of colors needed is at least the maximum vertex degree Δ. A theorem of Vizing's states that every graph can be properly edge colored using $\Delta + 1$ colors. Vizing's proof is constructive, and gives an algorithm to $\Delta + 1$ color a graph with n vertices and m edges in $O(mn)$ time (see [48]).

Our focus is on the edge coloring problem in the online setting, which was first introduced by Bar-Noy, Motwani and Naor [9]. Starting from the empty graph on vertex set V, edges of the graph are revealed one at a time, and the algorithm must irrevocably assign colors to the edges as they arrive. The simplest online edge coloring algorithms are greedy algorithms, which color each arriving edge by some previously used color whenever it is possible to do so. Since every arriving edge touches at most $2\Delta - 2$ existing edges, any greedy algorithm uses at most $2\Delta - 1$ colors. In their paper, [9] showed that for $\Delta = O(\log n)$, no deterministic online algorithm can guarantee better than $2\Delta - 1$ coloring and thus greedy algorithms are optimal in this case. They also showed that for $\Delta = O(\sqrt{\log n})$, no randomized online algorithm can achieve a better than $2\Delta - 1$ coloring. Furthermore, when $\Delta = \sqrt{n}$, the results of [29, 30] combine to show that no randomized online coloring algorithm can color using $\Delta + o(\sqrt{\Delta})$ colors. Thus, most of the focus has been designing algorithms for $\Delta = \omega(\log n)$ using $(1+o(1))\Delta$ colors. There has been steady progress on this problem [1, 15, 54, 46, 20], culminating in a recent work of [19], which presents a randomized algorithm that edge colors an online graph using $\Delta + o(\Delta)$ colors.

In this chapter, we investigate the *randomized greedy algorithm* which is a natural variation of the greedy algorithm. Given a set Γ of colors, the randomized greedy algorithm $\mathcal{A}$ on input the online graph G chooses the color of each arriving edge uniformly at random from the currently allowed colors for that edge, and leaves the edge uncolored if no colors are allowed. The algorithm is said to succeed if every edge is colored. The example in Figure 2.1 shows that the algorithm will fail if $|\Gamma| = \Delta + o(\sqrt{\Delta})$ (because the set of colors not used by the first $\Delta - 1$ edges is likely to be disjoint from the set of colors not used by the second $\Delta - 1$ edges.)

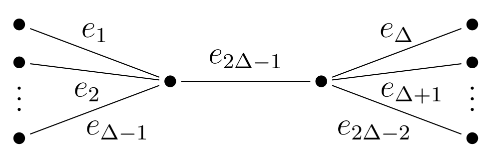

Figure 2.1: A simple example to illustrate that if $|\Gamma| = \Delta + o(\sqrt{\Delta})$, then $\mathcal{A}$ likely fails.

In their paper, [9] conjectured that given any online graph G with $\Delta = \omega(\log n)$ the randomized greedy algorithm $\mathcal{A}$ succeeds in $\Delta + o(\Delta)$ edge coloring G with high probability. This algorithm is simple and natural, but subsequent researchers (e.g. [46, 54, 2, 7]) have noted that it seems difficult to analyze. Prior to the present paper, the analysis was only done for the special case of trees (see Theorem 5 below). Our first result gives an analysis of this algorithm for the case of random-order arrivals. In this setting, an adversary can pick a worst-case graph but the edges of the graph arrive in a random order. (Recall that a graph with no multiple edges is said to be *simple*.)

Theorem 1 (Informal version of Theorem 6)**.** (**Random order case**) Let $\epsilon \in (0, 1)$ be a constant. The algorithm $\mathcal{A}$, when given any simple graph G of maximum degree $\Delta = \omega(\log n)$, whose edges are presented in a uniformly random order, edge colors G with $(1 + \epsilon)\Delta$ colors with high probability.

Our second result applies to the setting where the adversary is *adaptive*. This means that the adversary does not have to decide the graph *a priori*. They can instead decide the subsequent edges of graph based on the coloring of the prior edges. We call such a graph *adaptively chosen*.

Theorem 2 (Informal version of Theorem 7)**.** (**Dense case**) Let $\epsilon \in (0, 1)$, $M \geq 1$ be constants. Suppose G is an *adaptively chosen* simple graph with maximum degree Δ and $n \leq M\Delta$. Then, $\mathcal{A}$ succeeds in $(1 + \epsilon)\Delta$-edge coloring G with high probability.

To clarify the setting of Theorem 2, we note that prior results (as far as we know) assumed an *oblivious adversary* that chooses the graph G and the edge-arrival order but must fix *both* of these before their online algorithm receives any input. In contrast, an *adaptive* adversary builds the input as $\mathcal{A}$ runs and may choose each edge depending on the coloring of the graph so far.

For the case of online algorithms, [13] established a connection between randomized algorithms against an adaptive adversary and deterministic algorithms. Their result states that for a given online problem, if there is an α-competitive randomized algorithm against an adaptive adversary, then there exists a α-competitive deterministic algorithm. Exploiting this connection, we get the following corollary:

Corollary 3. Let $\epsilon \in (0, 1)$ be a constant. For Δ sufficiently large and $n = O(\Delta)$ there is a deterministic online algorithm that given any simple graph with n vertices and maximum degree Δ will produce a valid coloring using $(1 + \epsilon)\Delta$ colors.

Since the above-mentioned previous algorithms for online edge-coloring were applicable against an *oblivious adversary*, they did not have any implications to the deterministic setting. As far as we know, Corollary 3 is the first result for a general class of graphs that gives a deterministic online algorithm using fewer than $2\Delta - 1$ colors.

Related Work. The edge coloring problem has been considered in numerous settings. As mentioned before, Vizing's proof already lends itself to a polynomial time algorithm. Additionally, it is NP-hard to distinguish whether a graph is Δ or $\Delta + 1$ colorable [43]. Vizing's upper bound was subsequently improved to $O(m \cdot \min\left\{\Delta \cdot \log n, \sqrt{n \cdot \log n})\right\}$ by [3, 40] and then to $O(m \cdot \min\{\Delta \cdot \log n, \sqrt{n})\}$ by [56]. The current best known bounds are $O(n^2)$ for sparse graphs (due to [4]), and $O(mn^{-3})$ for dense graphs (due to [17]). There has also been a considerable effort to study edge coloring in other computational models: such as the dynamic model [10, 16, 32, 28, 18, 27], the distributed model [**ElkinK24**, 49, 37, 41, 8, 24, 14, 28, 31], and the streaming model [12, 11, 25, 42].

Remark 4 (Note about ϵ)**.** We only consider $\epsilon < 1$, since for $\epsilon \geq 1$, we have $\geq 2\Delta$ colors, and the randomized greedy algorithm must succeed.

2.1.1 Our Approach

We start with describing $\mathcal{A}$ in more detail. Fix the color set Γ. The edges of $G = (V, E)$ arrive in some order $e_1, e_2, ...$ and so on. At all points during the algorithm, for each vertex v, the *free set at* v is the set of colors not yet used to color an edge incident on v. The free set of v when edge e_i arrives is denoted $F_{i-1}(v)$. When edge $e_i = (u, v)$ arrives, algorithm $\mathcal{A}$ uniformly chooses a color c from $F_{i-1}(u) \cap F_{i-1}(v)$ and colors e_i with that color. If $F_{i-1}(u) \cap F_{i-1}(v) = \emptyset$ when e_i arrives then e_i is left uncolored and the algorithm continues. In this case the algorithm is said to *fail on* e_i. As mentioned in the introduction, this algorithm was previously analyzed for trees. We give a proof of this for completeness, since the tree case gives intuition for our proof strategy for general graphs.

Theorem 5. [36] Suppose the online graph G is a tree and $\Delta = \omega(\log n)$. If $|\Gamma| = \Delta + 2\sqrt{\Delta \cdot \log n}$, then $\mathcal{A}$ succeeds with probability at least $1 - 1/n$.

Proof. Let $|\Gamma| = (1+\epsilon)\cdot\Delta$, where $\epsilon \geq 2\sqrt{\frac{\log n}{\Delta}}$. Prior to the arrival of any edge $e_i = (u, v)$, the vertices u and v are in separate components of the tree, and therefore, $F_{i-1}(u)$ and $F_{i-1}(v)$ are independently sampled from Γ, even if we condition on not having failed in either component before e_i arrives. Conditioned on the algorithm having succeeded thus far, $k_1 = |F_{i-1}(u)|$ and $k_2 = |F_{i-1}(v)|$ are determined by the order of edges. By symmetry $F_{i-1}(u)$ and $F_{i-1}(v)$ are uniformly random from $\binom{\Gamma}{k_1}$, $\binom{\Gamma}{k_2}$, respectively. Conditioned on not having failed yet,

$$\begin{aligned}
\Pr\left[F_{i-1}(u) \cap F_{i-1}(v) = \emptyset\right] &= \sum_{S \in \binom{\Gamma}{k_1}} \Pr\left[F_{i-1}(u) \cap F_{i-1}(v) = \emptyset \mid F_{i-1}(u) = S\right] \cdot \Pr\left[F_{i-1}(u) = S\right] \\
&= \sum_{S \in \binom{\Gamma}{k_1}} \Pr\left[F_{i-1}(v) \cap S = \emptyset\right] \cdot \Pr\left[F_{i-1}(u) = S\right] \\
&\text{(Due to independence)} \\
&\leq \sum_{S \in \binom{\Gamma}{k_1}} \left(1 - \frac{k_1}{|\Gamma|}\right)^{k_2} \cdot \Pr\left[F_{i-1}(u) = S\right] \\
&\leq \exp\left(-\frac{k_1 k_2}{|\Gamma|}\right) \\
&\leq \exp\left(-\frac{\epsilon^2 \cdot \Delta}{(1+\epsilon)}\right) \\
&\left(\text{since } \epsilon = 2\sqrt{\frac{\log n}{\Delta}}\right) \\
&= O(n^{-3})
\end{aligned}$$

Taking a union bound over all edges e_i, $\mathcal{A}_\Gamma(G, \sigma)$ fails with probability $O(n^{-1})$. □

In order to analyze $\mathcal{A}$, we consider a modified algorithm, $\mathcal{A}'$ (see Definition 8), which produces a proper coloring with the same probability as $\mathcal{A}$. We first describe the intuition for $\mathcal{A}'$. For general graphs, when edge $e_i = (u, v)$ the $F_{i-1}(u)$ and $F_{i-1}(v)$ are in general not independent, but intuitively they should be approximately independent. Taking inspiration from the proof for trees, we aim to show that for any subset S of colors, the set $F_{i-1}(u)$ "looks random" with respect to S, i.e., that when edge e_i arrives, for any vertex w

$$|S \cap F_{i-1}(w)| \approx \frac{|S||F_{i-1}(w)|}{(1+\epsilon)\Delta}. \tag{2.1}$$

Note that the right hand side is the expected size of $|S \cap F_{i-1}(w)|$ if $F_{i-1}(w)$ was a uniformly random subset of Γ. It is fairly obvious that Equation (2.1) does not hold for every set S and every vertex w (for example, if $S = \Gamma - F_{i-1}(w)$). What we will show is that for each S, and for all but constantly many vertices v, Equation (2.1) holds for all time steps i (up to some small error).

The intuition for our proof is as follows. The set $F_i(v)$ evolves over time as colors are removed one by one. If, at each step j, the color removed was chosen uniformly from $F_{j-1}(v)$, then at any time i, the set $F_i(v)$ would be a uniform set from Γ and would likely satisfy Equation (2.1). In fact, since we are only interested in the *size* of $F_i(v) \cap S$ appearing to be random, and not the set itself, in order to satisfy Equation (2.1) with high probability, it is enough that the probability the color chosen at step j is in S is the same as it would be if the color was chosen uniformly from $F_{j-1}(v)$. More explicitly, we would like to have

$$\frac{|S \cap F_{j-1}(v) \cap F_{j-1}(w)|}{|F_{j-1}(v) \cap F_{j-1}(w)|} \approx \frac{|S \cap F_{j-1}(v)|}{|F_{j-1}(v)|}, \tag{2.2}$$

where the left hand side is the probability that the color selected for $e_j = (v, w_j)$ belongs to S. Note that if we set $S' = S \cap F_{j-1}(v)$ and $S'' = F_{j-1}(v)$, and in addition we knew that Equation (2.1) was satisfied for vertex w_j and each of the color sets S' and S'', then the above equation would hold (with a marginal increase in error.) This idea forms the basis of an inductive proof of our main lemma, Lemma 33.

As mentioned, we will show that with high probability, for each set S, the number of vertices v for which Equation (2.1) fails is bounded by a constant. Ideally, we would like to use this to show that for all but constantly many $e_j = (v, w_j)$ adjacent to v, Equation (2.2) holds. Unfortunately, it could happen that w_j is one of the vertices for which Equation (2.1) fails for $S' = S \cap F_{j-1}(v)$ or $S'' = F_{j-1}(v)$, and this would prevent us from reasoning as above. To get around this we note that the sets $F_j(v)$ cannot be completely independent from each other - $F_j(v)$ will differ from $F_{j+t}(v)$ by at most t colors. So for each vertex v, we will partition the time steps into a (large) constant number of v-*phases* so that each v-phase contains only a small fraction of the edges incident on v. Then for each v-phase r we will approximate $F_{j-1}(v)$ for all time steps j of that v-phase by the set $A_{j-1}(v)$ which is defined to be the free set of v at the beginning of the v-phase that contains j. Since $F_{j-1}(v)$ will not differ too much from $A_{j-1}(v)$, we will argue that if we replace $F_{j-1}(v)$ with $A_{j-1}(v)$ in Equation (2.2) and the new equation holds for all but constantly many $e_j = (v, w_j)$, then the original equation must also hold for all but constantly many neighbors w_j. This allows us to carry out the reasoning of the previous paragraph.

To this end, our analysis will consider a modified version of the algorithm which we denote by $\mathcal{A}'$. This modified version produces the same distribution over colorings as $\mathcal{A}$ but it has the advantage that it explicitly reflects the partition of time steps into v-phases for each v, and the approximation of $F_{j-1}(v)$ by $A_{j-1}(v)$ described above. The modified algorithm works as follows. When edge $e_j = (u, v)$ arrives we first sample a color uniformly from $A_{j-1}(u) \cap A_{j-1}(v)$ (rather than from $F_{j-1}(u) \cap F_{j-1}(v)$). If the selected color is valid (which means that it belongs to $F_{j-1}(u) \cap F_{j-1}(v)$) then we use it, otherwise we discard it and then sample uniformly from $F_{j-1}(u) \cap F_{j-1}(v)$.

We will have the following chain of comparisons: first, that the number of times $\mathcal{A}'$ initially selects an invalid color and resamples is small, and thus the colors chosen for $\mathcal{A}$ are close to the colors initially chosen by $\mathcal{A}'$. Second, the inductive assumption that Equation (2.2) holds for all but constantly many edges $e_j = (v, w_j)$ adjacent to v, and thus the colors initially chosen by $\mathcal{A}'$ hit any given set S about as often as a uniform choice from $F_{j-1}(v)$ would. Finally, we show that the uniform choice from $F_{j-1}(v)$ would likely satisfy Equation (2.1). Proposition 16 quantifies the error added in each step of this chain in terms of martingale sums defined in Section 2.3.1.

2.2 Preliminaries

2.2.1 Online Coloring

Online coloring can be described as a two player game between *Builder* and *Colorer*. The game $\Phi(n, \Delta, \epsilon)$ is parameterized by the number of vertices n, degree bound Δ and $\epsilon > 0$. The game starts with the empty graph on n vertices and a color set Γ of size $\lceil (1+\epsilon)\Delta \rceil$. The game lasts for $m = \lfloor \Delta n/2 \rfloor$ steps. In each time step, Builder selects an edge to add to the graph, subject to the restriction that all vertex degrees remain below Δ. Colorer then assigns a color to the edge from Γ so that the overall coloring remains valid. If this is impossible (i.e. any color choice will invalidate the coloring) the edge is left uncolored. Colorer wins if every edge is successfully colored. For convenience, we also allow Builder to add *null edges*; such edges are not adjacent to any vertex in the graph and may be assigned any color without affecting whether or not the coloring is valid. This has the advantage that we can fix the number of steps to m, and if Builder gets stuck (is unable to add an edge without violating the degree bound) then he can add null edges for the remaining steps.

The *state* of the game after i steps, denoted $\mathcal{S}_i$, consists of the list of the first i edges chosen by Builder and the coloring chosen by Colorer. A strategy for Builder is a function which given any game state $\mathcal{S}_i$ determines the next edge to be added or terminates the game. A strategy for Colorer is a function which given the game state $\mathcal{S}_i$ and an additional edge e assigns a color to e (or leaves e uncolored if no color can be assigned.) When Colorer's strategy is randomized and Builder's strategy is fixed, the game state becomes a random variable. Henceforth, when we discuss the (partial) "coloring" produced by $\mathcal{A}'$ after step i, we will be referring to the game state $\mathcal{S}_i$.

We say that a color c is *free for v at step i* if among the first $i-1$ edges, no edge that touches v is colored by c. If the ith edge chosen by Builder is (u, v) then the set of colors that can be used to color (u, v) is the intersection of the free set for v at step i and the free set for u at step i. If that set is empty then the edge is necessarily left uncolored.

We are interested in analyzing the behavior of the *randomized greedy strategy* for Colorer, denoted $\mathcal{A}$: for each new edge $e_i = (u, v)$, if the intersection of the free set for u and the free set for v is nonempty then choose the color for e_i uniformly at random from that set.

During the game Builder produces a graph G together with an ordering of its edges which we view as a one-to-one function $\sigma : E(G) \longrightarrow [m]$. Elements in $[m]$ to which no edge is in $E(G)$ is mapped are interpreted as *null edges*. We refer to (G, σ) as an edge-ordered graph. In general the edge ordered graph produced by Builder may depend on the coloring of edges chosen by Colorer. A strategy of Builder is *oblivious* if the choice of edge to be added at each step depends only on the current edge set and not on the coloring. An oblivous strategy is fully described by the pair (G, σ) where the edge selected at step i is σ_i^{-1} (and is a null edge if that is undefined). We denote this strategy by $\mathbf{obl}(G, \sigma)$. We can now give a more precise formulation of our first main result.

Theorem 6. (**random order case**) For any constant $\epsilon \in (0, 1)$ there are constants $N = N(\epsilon)$ (sufficiently large), $\gamma_1 = \gamma_1(\epsilon)$ and $\gamma_2 = \gamma_2(\epsilon)$ (sufficiently small) such that the following holds. Suppose that n is sufficiently large, $\Delta > N\log(n)$ and consider the edge coloring game $\Phi(n, \Delta, \epsilon)$. For any G on n vertices with maximum degree Δ, for all but at most a $2^{-\gamma_1\Delta}$ fraction of mappings σ of $E(G)$ to $[m]$, $\mathcal{A}$ will defeat the oblivious strategy $\mathbf{obl}(G, \sigma)$ (i.e., produce an edge coloring for G) with probability at least $1 - 2^{-\gamma_2\Delta}$.

Our second main result applies to arbitrary adaptive strategies of Builder. (Here we use the word *adaptive* to emphasize that Builder's choices may depend on the past coloring.)

Theorem 7. (**dense case**) For any constant $\epsilon \in (0, 1)$ and constant $M > 1$, there is a constant $\gamma = \gamma(\epsilon, M)$ so that the following holds. For sufficiently large n, and for $\Delta > n/M$, for any (possibly adaptive) strategy for the online coloring game $\Phi(n, \Delta, \epsilon)$, $\mathcal{A}$ wins (produces an edge coloring of the resulting graph) with probability $1 - 2^{-\gamma\Delta}$.

2.2.2 The Algorithm $\mathcal{A}'$

In order to prove the above theorems we analyze a modified strategy $\mathcal{A}'$ that against any given Builder strategy produces exactly the same distribution over colorings as $\mathcal{A}$, but will be easier to analyze.

Phase Counter Functions. For each vertex v, the modified algorithm will keep track of a partition of the time steps into at most b contiguous v-*phases*. The value of b is specified in Section 2.3.2. For each v, the v-phases are numbered from 1 to b. The partition into v-phases is represented by a *phase-partition counter* $\{\phi_i(v) : i \in \{1, \dots, t\}\}$ where $\phi_i(v)$ is the number of the v-phase that contains time step i. For each $i > 1$ we have either $\phi_i(v) = \phi_{i-1}(v)$ (if $i - 1$ and i are in the same phase) or $\phi_i(v) = 1 + \phi_{i-1}(v)$ (if i starts a new v-phase.) This function is determined online, so that $\phi_i(v)$ is determined after step $i - 1$ of the game.

The phase-partition counters that we use are defined formally in Definition 27. In the dense case (Theorem 7) for each vertex v, the v-phases are determined by the number of edges incident to v that have arrived. For $r \geq 2$, the r^{th} v-phase starts with the time step where the number of edges incident on v first exceeds $(r-1)\Delta/b$. (Thus the number of edges incident on v in each v-phase is within 1 of Δ/b.) The phase-partition counters in this case are denoted by ϕ^D.

In the random case (Theorem 6), the v-phase partition is the same for every vertex. For $r \geq 2$, the r^{th} v-phase starts with the time step where the total number of edges arrived first exceeds $(r-1)m/b$. The phase-partition counters in this case are denoted ϕ^R.

Algorithm Description In $\mathcal{A}'$, Colorer maintains for each vertex v, a color set $A_i(v)$ that approximates $F_i(v)$ but remains constant during each v-phase. We call $A_i(v)$ the *palette* of v at the end of time step i. For vertex v and time i, we define $A_{i-1}(v) = \Gamma$ if i belongs to the first v-phase and otherwise:

$$
\begin{aligned}
A_{i-1}(v) &= F_{i'-1}(v) \text{ where } i' \text{ is the first step of the } v\text{-phase containing step } i. \\
&= \text{the set of available (free) colors } v \text{ at the start of } v\text{-phase } \phi_i(v).
\end{aligned}
$$

Note that $A_i(v) \supset F_i(v)$ for all i, and we think of $A_i(v)$ as an (over-)approximation to $F_i(v)$.

Definition 8 (Algorithm $\mathcal{A}'$)**.** Start with $A_0(v) = F_0(v) = \Gamma$ for all v. When edge $e_i = (u, v)$ arrives:

(a) Choose c uniformly at random from $A_{i-1}(u) \cap A_{i-1}(v)$. This is the *preliminary color for* e_i. If $A_{i-1}(u) \cap A_{i-1}(v) = \emptyset$, e_i is left uncolored.

(b) Next, choose the final color for e_i:

  i) If $c \in F_{i-1}(u) \cap F_{i-1}(v)$, color edge e_i with c.

ii) Otherwise, $c \notin F_{i-1}(u) \cap F_{i-1}(v)$. (We refer to this as a *collision* at e.) In this case, choose c' uniformly from $F_{i-1}(u) \cap F_{i-1}(v)$ for edge e_i. If $F_{i-1}(u) \cap F_{i-1}(v) = \emptyset$, e_i is left uncolored.

(c) For all vertices w, if $\phi_i(w) < \phi_{i+1}(w)$ (i completes the current w-phase) then $A_i(w)$ is set to $F_i(w)$, otherwise $A_i(w) = A_{i-1}(w)$.

It is obvious from the algorithm that the final color selected for e is uniformly random over $F_{i-1}(u) \cap F_{i-1}(v)$, so the distribution over colorings produced by $\mathcal{A}'$ is the same as $\mathcal{A}$. To prove Theorems 7 and 6 it suffices to prove the corresponding statements with $\mathcal{A}$ replaced by $\mathcal{A}'$ and this is what we'll do.

2.2.3 Framework

Here we provide some necessary background information and notation, as well as an overview of the lens through which we view the algorithm. This framework will inform much of the notation and intuition in the succeeding sections, and also provide a foundation for extensions of the main result.

A discrete probability space $(\Omega, \mathbb{P})$ consists of a countable set of outcomes, Ω, and a probability measure $\mathbb{P} : \Omega \to [0,1]$ such that $\sum\limits_{\omega \in \Omega} \mathbb{P}(\omega) = 1$. In our case, Ω will be the space of game states resulting from algorithm $\mathcal{A}'$ on G and $\mathbb{P}$ is the probability of each outcome. For a discrete probability space, a filtration, $\{\mathcal{F}_i\}_{i=0}^m$, is a sequence of partitions of Ω where $\mathcal{F}_0$ is the trivial partition with one part, and each $\mathcal{F}_{i+1}$ is a refinement of $\mathcal{F}_i$. For analysis purposes we augment $\mathcal{S}_i$ so that it includes the preliminary color that $\mathcal{A}'$ chooses for each edge, and let $\mathcal{F}_i$ correspond to the set of possible states for $\mathcal{S}_i$, i.e., it is the partition of the probability space into sets that agree on the i^{th} state. We will say that a random variable $Q : \Omega \to \mathbb{R}$ depends only on $\mathcal{F}_i$ if, for each set in the partition, Q is constant on that set (so it is determined by the state $\mathcal{S}_i$.)

Notation. For a step $1 \leq i \leq m$ we define the following random variables depending only on $\mathcal{F}_i$:

$$e_{i+1} \quad = \quad \text{the } (i+1)^{th} \text{ edge to arrive.}$$

For vertex v and time step i, we define the following random variables that are determined by $\mathcal{F}_i$:

$$\begin{aligned} F_i(v) &= \text{the set of free colors of } v \text{ at the end of step } i. \\ \phi_{i+1}(v) &= \text{the } v\text{-phase for step } i+1. \\ A_i(v) &= \text{the palette of free colors of } v \text{ at the end of step } i. \end{aligned}$$

We will track the above variables as the algorithm progresses, and they will form the basis for the martingale difference sequences we analyze in Section 2.3. Each of these variables is subscripted by a step i, at which time they are fixed. This is what allows us to apply our concentration lemmas and make probabilistic claims about such variables.

In contrast, the variables defined below will be superscripted by a phase number r, if at all, and are defined only after the last edge in phase r of v arrives and is colored (or fails to be colored.) However, since the choice of edges is not fixed ahead of time, we do not know a priori the step i at which they will be defined. Thus, we must be careful when dealing with such quantities. In the rest of the paper, outside of Section 2.3, we will assume the algorithm has concluded, and make deterministic statements about the quantities defined below, conditioned on the results from Section 2.3.

For a vertex v the following set depends on the entire outcome, $\mathcal{F}_m$:

$$T(v) \quad = \quad \text{the set of arrival times of edges adjacent to } v.$$

Also for vertex v and v-phase r, we define the following sets depending on $\mathcal{F}_m$:

$$\begin{aligned}
A^r(v) &= \text{the set of free colors of } v \text{ at the end of the } r^{th} \text{ } v\text{-phase.} \\
U^r(v) &= A^{r-1}(v) \setminus A^r(v) \\
&= \text{ the set of colors used to color edges incident on } v \text{ during the } r^{th} \text{ } v\text{-phase.} \\
T^r(v) &= \{i \in T(v) : \phi_i(v) = r\} \\
&= \text{ the set of arrival times of edges incident on } v \text{ during the } r^{th} \text{ } v\text{-phase.} \\
T^{\leq r}(v) &= \cup_{r' \leq r} T^{r'}(v). \\
\mathbf{last}^r(v) &= \max\{i : i \in T^r(v)\} \\
&= \text{ the time at which the last edge of } v\text{-phase } r \text{ that is adjacent to } v \text{ arrives.}
\end{aligned}$$

We remark that by this notation, for each time i, $A_{i-1}(v) = A^{\phi_i(v)-1}(v)$.

Finally, we define the following quantity, called the *error of vertex v with respect to S after phase r*:

$$\delta^r(v, S) := \frac{|A^r(v) \cap S|}{|A^r(v)|} - \frac{|S|}{(1+\epsilon)\Delta}. \tag{2.3}$$

In our inductive arguments, we will be proving statements about quantities of the form $Q^r(v)$ based on the relative ordering of the times $\mathbf{last}^r(v)$. Thus, the times $\mathbf{last}^r(v)$ are important to note down. More formally, we define the following partial order on vertex-phase pairs $(v, r) \in V \times [b]$.

Definition 9 (Vertex-Phase and Edge Arrival Ordering, $\prec$)**.** Given pairs $(v, r), (u, s) \in V(G) \times [b]$, we say that $(v, r) \prec (u, s)$ if $\mathbf{last}^r(v) < \mathbf{last}^s(u)$.

We make one final note about the variables defined in this section. Above we have defined two families of variables: those indexed by times steps i that are defined at a known time step i, and those that are not defined until the algorithms concludes. Note that although the sets in the second family depend on $\mathcal{F}_m$, the indicator variable for a particular index i belonging to one of these sets only depends on $\mathcal{F}_{i-1}$, since the identity of e_i and $\phi_i(v)$ are determined at that point. For instance, $T^r(v)$ depends on $\mathcal{F}_m$, but the indicator $\mathbb{1}_{\{i \in T^r(v)\}}$ only depends on $\mathcal{F}_{i-1}$. Thus, indicator variables of this form will also belong to the first family of variables and will be used in Section 2.3.

2.2.4 Proof Idea

As mentioned above, for each set $S \subseteq \Gamma$ and vertex $v \in V$, we would like to track $|F_i(v) \cap S|$ as $\mathcal{A}'$ progresses. We will do this indirectly by tracking $\frac{|A^r(v) \cap S|}{|A^r(v)|}$ with the goal of bounding $\delta^r(v, S)$. Observe that $\delta^0(v, S) = 0$, so our goal will be to bound how much $\delta^r(v, S)$ increases in each phase. Note that if $U^r(v)$ was chosen uniformly from $A^{r-1}(v)$, we would expect that

$$|U^r(v) \cap S| \approx |U^r(v)| \cdot \frac{|A^{r-1}(v) \cap S|}{|A^{r-1}(v)|}, \tag{2.4}$$

which would imply that the error with respect to any set S would not increase too much when the phase of v changes:

$$\begin{aligned} \delta^{r-1}(v, S) - \delta^r(v, S) &= \frac{|A^{r-1}(v) \cap S|}{|A^{r-1}(v)|} - \frac{|A^r(v) \cap S|}{|A^r(v)|} \\ &= \frac{|A^{r-1}(v) \cap S|}{|A^{r-1}(v)|} - \frac{|A^{r-1}(v) \cap S| - |U^r(v) \cap S|}{|A^r(v)|} \\ &= \frac{1}{|A^r(v)|} \left(|U^r(v) \cap S| + |A^{r-1}(v) \cap S| \left(\frac{|A^r(v)|}{|A^{r-1}(v)|} - 1 \right) \right) \end{aligned} \tag{2.5}$$

$$= \frac{1}{|A^r(v)|}\left(|U^r(v)\cap S| + |A^{r-1}(v)\cap S|\cdot\frac{|A^r(v)|-|A^{r-1}(v)|}{|A^{r-1}(v)|}\right)$$
$$= \frac{1}{|A^r(v)|}\left(|U^r(v)\cap S| - |A^{r-1}(v)\cap S|\cdot\frac{|U^r(v)|}{|A^{r-1}(v)|}\right). \tag{2.6}$$

Our goal is to show that Equation (2.4) is never too far from the truth, and therefore $\delta^r(v,S)$ does not grow too large in any given phase. As we previously noted, this cannot hold for every set S and every vertex v, but we can show that there exists a constant C depending on ϵ such that with high probability, for each set S, for all but at most C vertices v and all phases r, we have

$$|\delta^r(v,S)| \le \frac{\epsilon^3\Delta}{10|A^r(v)|}. \tag{2.7}$$

Proposition 16 allows us to bound the amount $\delta^r(v,S)$ grows during a phase of v in terms of three main sources of error: the error from the collisions at each phase, the error inherent to a locally independent algorithm, and the error in the palettes of the neighbors of v for that phase.

To that end, for all $e_i=(u,v)$, we define the indicator variables $X_i(S)$ to track whether the preliminary color chosen for e_i from $A_{i-1}(u)\cap A_{i-1}(v)$ hits S, $Y_i(S)$ to track whether the final color chosen for e_i from $A_{i-1}(u)\cap A_{i-1}(v)$ hits S, and the collision indicator variables Z_i to track whether the preliminary color chosen for e_i needs to be resampled. Furthermore, we let

$$p_i(S) := \frac{|A_{i-1}(u)\cap A_{i-1}(v)\cap S|}{|A_{i-1}(u)\cap A_{i-1}(v)|} \tag{2.8}$$

be the probability that $X_i(S)=1$, conditioned on the partial coloring of edges before e_i arrives, and let

$$D_i(S) = X_i(S) - p_i(S). \tag{2.9}$$

Note that,

$$\left| |U^r(v) \cap S| - \sum_{j \in T^r(v)} X_j(S) \right| = \left| \sum_{j \in T^r(v)} Y_j(S) - X_j(S) \right| \leq \sum_{j \in T^r(v)} Z_j \tag{2.10}$$

since the colors used in the preliminary coloring and the final coloring differ only in the edges which experience collisions. This produces the first source of error. Similarly,

$$\left| \sum_{j \in T^r(v)} X_j(S) - \sum_{j \in T^r(v)} p_j(S) \right| = \left| \sum_{j \in T^r(v)} D_j(S) \right| \tag{2.11}$$

models the inherent error of the local algorithm on phase r of v. Therefore, if we can show that for most $i \in T^r(v)$,

$$p_i(S) \approx \frac{|A^{r-1}(v) \cap S|}{|A^{r-1}(v)|} \tag{2.12}$$

and bound the quantities on the right hand sides of Equation (2.10) and Equation (2.11), we will have

$$|U^{r+1}(v) \cap S| \approx |U^{r+1}(v)| \cdot \frac{|A^r(v) \cap S|}{|A^r(v)|}, \tag{2.13}$$

as desired. Our main tool for bounding the sums above will be martingale concentration inequalities.

2.2.5 Background on Martingales

In this section we review needed definitions and facts about martingales including Freedman's concentration inequality. We also use Freedman's inequality to deduce a more general concentration lemma.

A martingale with respect to $\{\mathcal{F}_i\}$ is a sequence $\{Y_i\}$ of random variables such that Y_i depends only on $\mathcal{F}_i$, and

$$\mathbb{E}[Y_i \mid \mathcal{F}_{i-1}] = Y_{i-1}.$$

Similarly, a supermartingale is a sequence $\{Y_i\}$ such that

$$\mathbb{E}[Y_i \mid \mathcal{F}_{i-1}] \leq Y_{i-1}$$

and a submartingale is a sequence $\{Y_i\}$ such that

$$\mathbb{E}[Y_i \mid \mathcal{F}_{i-1}] \geq Y_{i-1}.$$

Note that a martingale is both a supermartingale and a submartingale. We can also consider a martingale difference sequence, which is a sequence $\{D_i\}$ satisfying

$$\mathbb{E}[D_i \mid \mathcal{F}_{i-1}] = 0.$$

For any difference sequence $\{D_i\}$, the sums $\{Y_i = \sum_{j=1}^{i} D_j\}$ form a martingale and for any martingale $\{Y_i\}$, the differences $\{D_i = Y_i - Y_{i-1}\}$ form a martingale difference sequence. We make the following observation about properties of difference sequences that will be useful later on:

Observation 10. Let $\{D_i\}$ be a martingale difference sequence with respect to $\{\mathcal{F}_i\}$.

(a) For any $\ell < i$, conditioning on $\mathcal{F}_{i-1}$ fixes D_ℓ, so we have

$$\mathbb{E}[D_\ell D_i \mid \mathcal{F}_{i-1}] = D_\ell(\mathcal{F}_{i-1})\, \mathbb{E}[D_i \mid \mathcal{F}_{i-1}] = 0.$$

(b) More generally, let $\{\beta_i\}$ be a sequence of random variables where β_i is determined by $\mathcal{F}_{i-1}$. Then the sequence $\{\beta_i D_i\}$ is also a martingale difference sequence with respect to $\{\mathcal{F}_i\}$:

$$\mathbb{E}[\beta_i D_i \mid \mathcal{F}_{i-1}] = \beta_i(\mathcal{F}_{i-1})\,\mathbb{E}[D_i \mid \mathcal{F}_{i-1}] = 0.$$

We say that $\{\beta_i D_i\}$ is *derived from* the martingale $\{D_i\}$ and refer to β_i as the *coefficient sequence* for the derived martingale. We emphasize that the coefficients here are themselves random variables. In this chapter, the difference sequences that we consider are indexed by the time steps of the process (corresponding to the edges of the graph) and we will associate a β sequence to each vertex, where the β-sequence associated to v is nonzero only on the edges that touch v.

A key property of martingales is that if the elements of the difference sequence of a martingale are bounded, then with high probability, the martingale does not deviate too far from its starting value. This theorem is formally stated below.

Lemma 11 (Azuma-Hoeffding Bounds)**.** Suppose $\{Y_i\}$ is a supermartingale relative to some difference sequence $\{D_i\}_{i\geq 1}$, whose increments $D_i = Y_i - Y_{i-1}$ satisfy $|D_i| \leq \sigma_i$. Then,

$$\Pr\left[Y_n \geq Y_0 + \delta\right] \leq \exp\left(\frac{-\delta^2}{2\sum_{i=1}^n \sigma_i^2}\right).$$

Similarly, suppose $\{Y_i\}$ is a submartingale relative to some sequence $\{D_i\}_{i\geq 1}$ whose increments $D_i = Y_i - Y_{i-1}$ satisfy $|D_i| \leq \sigma_i$. Then,

$$\Pr\left[Y_n \leq Y_0 - \delta\right] \leq \exp\left(\frac{-\delta^2}{2\sum_{i=1}^n \sigma_i^2}\right).$$

Unfortunately, such a bound will not be sufficient for our purposes, since randomness in the choice of edges renders us unable to bound any particular difference D_i, despite knowing that most of these values will be zero. Instead we use a variant of

Azuma-Hoeffding due to Freedman that is well-suited for analyzing processes whose evolution is partially controlled by an adversary (which for us is Builder). In Freedman's theorem one considers the auxiliary sequence $V_i = \mathrm{Var}(D_i \mid \mathcal{F}_{i-1})$, which is the variance of D_i conditioned on $\mathcal{F}_{i-1}$).

Lemma 12. [39, Theorem 4.1] Suppose $\{Y_i\}$ is a supermartingale with respect to $\{\mathcal{F}_i\}$ and its corresponding difference sequence $\{D_i\}$ satisfies $|D_i| \le D$ for all i. Let $V_i = \mathrm{Var}(D_i \mid \mathcal{F}_{i-1})$ and $W_i = \sum_{j \le i} V_j$, and suppose $W_m \le b$ with probability 1. Then

$$\Pr\left[Y_m \ge Y_0 + \delta\right] \le \exp\left(-\frac{\delta^2}{2(D \cdot \delta/3 + b)}\right).$$

In the next section we will use Lemma 12 to bound the probability that the derived supermartingale associated to a vertex gets too large. This bound is stated in the first part of the Lemma below. Additionally, there will be times we want to show that for a set of C vertices, $v_1, ..., v_C$, the derived supermartingales associated to each of those vertices cannot *all* become too large at once. In particular, we would like to show that the probability of this occurring decays exponentially in C. Note that if we were guaranteed that the edges adjacent to each of the v_k were disjoint and arrived contiguously - that is, if we could partition the difference sequence $\{D_i\}_{i=1}^{m}$ into C sequences $\{D_i\}_{i=1}^{m_1}, ..., \{D_i\}_{i=m_{C-1}+1}^{m_C}$ such that the derived supermartingale associated to v_k was nonzero only on $\{D_i\}_{i=m_{k-1}+1}^{m_k}$ - then we could get this result by iteratively applying Lemma 12, since conditioned on the value of $\{D_i\}_{i=1}^{m_{k-1}}$, the sequence $\{D_i\}_{i=m_{k-1}+1}^{m_k}$ is still a difference sequence. However, in our case, the arrival times of the edges for the different vertices can be interleaved, and even intersect. Nevertheless, we manage to provide a general sufficient condition for a similar conclusion to hold.

Lemma 13. Suppose $\{D_i\}_{i=1}^{t}$ is a martingale difference sequence with respect to $\{\mathcal{F}_i\}$ such that $|D_i| \le 1$ for all i and let α, a, Δ be positive real numbers where Δ is sufficiently large (depending on a and α).

(a) Suppose $\{\beta_i\}_{i=1}^t$ is a coefficient sequence where β_i depends only on F_{i-1} and such that with probability 1, $\sum_{i=1}^t |\beta_i| \leq \Delta$ and $|\beta_i| \leq a$ for all i. Then:

$$\Pr\left[\sum_i \beta_i D_i \geq \alpha\Delta\right] \leq \exp\left(-\frac{\alpha^4\Delta}{128a}\right).$$

(b) Suppose that for each $k \in \{1, \ldots, C\}$, $\{\beta_i^k\}_{i=1}^t$ is a coefficient sequence where β_i^k depends only on F_{i-1} and such that with probability 1, $\sum_{i=1}^t |\beta_i^k| \leq \Delta$ for all k. Suppose further that with probability 1, for all $i \in \{1, \ldots, t\}$, $\sum_{k=1}^C |\beta_i^k| \leq a$. Then:

$$\Pr\left[\forall k, \sum_i \beta_i^k D_i \geq \alpha\Delta\right] \leq \exp\left(-\frac{C\alpha^4\Delta}{128a}\right).$$

The key thing to note about the conclusion is that for α, Δ and a fixed, the probability upper bound shrinks exponentially with C. The first part of the Lemma is just the case $C = 1$ of the second part; we stated it separately to help the reader to digest the lemma statement, and also because the special case $C = 1$ will be applied twice in what follows.

Proof of Lemma 13. The proof is obtained by applying Lemma 12 to a single random sequence $\{Y_j\}$ that is constructed from $\{D_i\}$ and all C coefficient sequences. Let $Y_0 = 0$, and $j \in \{1, \ldots, t\}$ let:

$$Y_j = \sum_{k=1}^C \left[\left(\sum_{i \leq j} \beta_i^k D_i\right)^2 - \sum_{i \leq j} (\beta_i^k)^2\right].$$

If it is the case that for all k, $|\sum_{i=1}^t \beta_i^k D_i| \geq \alpha\Delta$ then:

$$Y_t = \sum_{k=1}^C \left[\left(\sum_{i \leq t} \beta_i^k D_i\right)^2 - \sum_{i \leq t} (\beta_i^k)^2\right] \geq C\alpha^2\Delta^2 - C\Delta a \geq \frac{C\alpha^2\Delta^2}{2},$$

for Δ sufficiently large and therefore:

$$\Pr\left[\forall k, \sum_i \beta_i^k D_i \geq \alpha\Delta\right] \leq \Pr\left[Y_t \geq \frac{C\alpha^2\Delta^2}{2}\right],$$

so it suffices to bound the probability on the right. We first show that $\{Y_j\}$ is a supermartingale. Defining $\{Z_j\}$ to be the difference sequence associated to $\{Y_j\}$, we have

$$Z_j = \sum_{k=1}^{C}\left[(\beta_j^k)^2 D_j^2 + 2(\beta_j^k) D_j \sum_{i<j} \beta_i^k D_i - (\beta_j^k)^2\right].$$

To see that $\{Y_j\}$ is a supermartingale, note that by Observation 10, for any $\ell < i$,

$$\mathbb{E}[D_\ell D_i \mid \mathcal{F}_{i-1}] = D_\ell\, \mathbb{E}[D_i \mid \mathcal{F}_{i-1}] = 0.$$

Furthermore, since $|D_i| \leq 1$ for all i, we have $\mathbb{E}[D_i^2 \mid \mathcal{F}_{i-1}] \leq 1$. Therefore, for any $1 \leq j \leq t$, we have

$$\begin{aligned}
\mathbb{E}[Y_j - Y_{j-1} \mid \mathcal{F}_{j-1}] = \mathbb{E}[Z_j \mid \mathcal{F}_{j-1}] &= \sum_{k=1}^{C}\left[(\beta_j^k)^2\, \mathbb{E}[D_j^2 \mid \mathcal{F}_{j-1}] + 2\beta_j^k \sum_{i<j} \beta_i^k\, \mathbb{E}[D_j D_i \mid \mathcal{F}_{j-1}] - (\beta_j^k)^2\right] \\
&= \sum_{k=1}^{C}(\beta_j^k)^2\left(\mathbb{E}[D_j^2 \mid \mathcal{F}_{j-1}] - 1\right) \\
&\leq 0,
\end{aligned}$$

which shows that $\{Y_j\}$ is indeed a supermartingale. We now will use Lemma 12 to upper bound the indicated probability. For this, we must bound the variance sums $\{W_j\}$ of $\{Y_j\}$ and the absolute values of the associated difference sequences $\{Z_i\}$. Note:

$$|Z_j| \leq \sum_{k=1}^{C}\left|(\beta_j^k)^2 D_j^2 + 2\beta_j^k D_j \sum_{i<j} \beta_i^k D_i - (\beta_j^k)^2\right|$$

$$
\begin{aligned}
&\le \sum_{k=1}^{C} \left(2|\beta_j^k|^2 + 2|\beta_j^k| \sum_{i<j} |\beta_i^k| \right) && \text{(since } |D_i| \le 1\text{)} \\
&= \sum_{k=1}^{C} 2|\beta_j^k| \left(\sum_{i \le j} |\beta_i^k| \right) \\
&\le 2\Delta \sum_{k=1}^{C} |\beta_j^k| && \text{(since } \sum_i |\beta_i^k| \le \Delta\text{)} \\
&\le 2\Delta a && \text{(since } \sum_k |\beta_i^k| \le a\text{)}
\end{aligned}
$$

Thus,

$$V_j = \mathrm{Var}(Z_j \mid \mathcal{F}_{j-1}) \le \mathbb{E}[Z_j^2 \mid \mathcal{F}_{j-1}] \le 2\Delta a \sum_{k=1}^{C} 4\Delta|\beta_j^k| = 8\Delta^2 a \sum_{k=1}^{C} |\beta_j^k|$$

which tells us

$$W_t \le \sum_j V_j \le \sum_j 8\Delta^2 a \sum_{k=1}^{C} |\beta_j^k| = 8\Delta^2 a \sum_{k=1}^{C} \sum_j |\beta_j^k| \le 8\Delta^2 a \sum_{k=1}^{C} \Delta \le 8aC\Delta^3$$

Then Lemma 12 tells us that,

$$
\begin{aligned}
\Pr\left[Y_t \ge \frac{C\alpha^2\Delta^2}{2} \right] &\le \exp\left(-\frac{C^2\alpha^4\Delta^4}{4 \cdot (C\alpha^2 a\Delta^3 + 16aC\Delta^3)} \right) \\
&\le \exp\left(-\frac{C\alpha^4\Delta}{128a} \right).
\end{aligned}
$$

□

2.3 Well-Behaved Colorings

2.3.1 Some Martingales Difference Sequences

Our main goal in this section will be to define the martingale difference sequences we will be considering. Recall that we are viewing the progression of the algorithm as a filtered probability space with $\mathcal{F}_i$ representing the space of partial colorings of the first i edges to arrive. We first introduce the random variables which will form the basis for the difference sequences we track throughout the course of the algorithm. All of the quantities defined below for an edge e_i will be set to 0 by default if e_i is null or we are unable to color e_i.

Definition 14 (Collision Variables)**.** The following random variables relate to the collisions experienced by algorithm $\mathcal{A}'$. For a non-null edge $e_i = (u, v)$

- Z_i is defined to be 1 if e_i is in a collision and then successfully colored, and is 0 otherwise. Thus $Z_i = 1$ provided that there is a collision at e_i (i.e., the preliminary color is not valid) and $F_{i-1}(u) \cap F_{i-1}(v) \neq \emptyset$.

- $q_i = \mathbb{E}[Z_i \mid \mathcal{F}_{i-1}]$. If $F_{i-1}(u) \cap F_{i-1}(v) = \emptyset$ this is 0. Otherwise:

$$q_i := 1 - \frac{|F_{i-1}(u) \cap F_{i-1}(v)|}{|A_{i-1}(u) \cap A_{i-1}(v)|} = \frac{|(A_{i-1}(u) \cap A_{i-1}(v)) \setminus (F_{i-1}(u) \cap F_{i-1}(v))|}{|A_{i-1}(u) \cap A_{i-1}(v)|}.$$

- $\{Z_i - q_i\}_{i=1}^m$ is a martingale difference sequence with respect to $\{\mathcal{F}_i\}$, since $\mathbb{E}[Z_i - q_i \mid \mathcal{F}_{i-1}] = 0$. Additionally, since $Z_i, q_i \in [0, 1]$, we have for all i,

$$|Z_i - q_i| \leq 1.$$

The significance of the next set of variables is little more subtle. Recall that our goal is to bound

$$\delta^r(v, S) := \frac{|A^r(v) \cap S|}{|A^r(v)|} - \frac{|S|}{(1+\epsilon)\Delta},$$

the error of vertex v with respect to color set S after its r^{th} phase. A natural way to do this would be to track how often the colors chosen for edges incident to v hit S. However, the probability that the color of an edge (u, v) hits S is highly dependent on the palette of u, which makes it difficult to control $\delta^r(v, S)$ on its own. Instead we consider a related, and easier to control, quantity: the difference between the probability of the color of an edge (u, v) hitting S and the indicator for the event. This doesn't directly bound $\delta^r(v, S)$, but it does allow us to approximate $|A^r(v) \cap S|$ in terms of intersections of the form $|A^{r'}(v') \cap S'|$ for neighbors v' of v which - crucially - complete their phase r' before v completes its phase r. This will allow us to use an inductive argument to bound $\delta^r(v, S)$ in terms of such $\delta^{r'}(v', S')$.

Definition 15 (Difference Variables)**.** Let $e_i = (u, v)$ be a non-null edge and $S \subseteq \Gamma$.

- $X_i(S)$ is 1 if the preliminary color for e_i belongs to S and 0 otherwise.
- $Y_i(S)$ is 1 if the final color chosen for e_i belongs to S and is 0 otherwise. Note that

  $$|X_i(S) - Y_i(S)| \leq Z_i,$$

  since the final color chosen for e_i differs from the preliminary color only if there is a collision.
- $p_i(S)$ is the probability that the preliminary color chosen for edge e_i is in S, conditioned on the coloring of all previous edges:

  $$p_i(S) := \mathbb{E}[X_i(S) \mid \mathcal{F}_{i-1}] = \frac{|A_{i-1}(u) \cap A_{i-1}(v) \cap S|}{|A_{i-1}(u) \cap A_{i-1}(v)|}.$$

- $D_i(S) = X_i(S) - p_i(S)$. This is a martingale difference sequence with respect to $\mathcal{F}_i$ since:

  $$\mathbb{E}[D_i(S) \mid \mathcal{F}_{i-1}] = \mathbb{E}[X_i(S) - p_i(S) \mid \mathcal{F}_{i-1}] = 0,$$

- Furthermore if $S_1, S_2, \ldots$ is a sequence of color sets where S_i is determined by $\mathcal{F}_{i-1}$ then $\{D_i(S_i)\}$ is also a martingale difference sequence, satisfying

  $$|D_i(S_i)| = |X_i(S_i) - p_i(S_i)| \leq 1$$

for all i.

The following proposition relates the variables above to the error $\delta^r(v,S)$ and motivates the difference sequences we will define. For a vertex v and phase ℓ, let

$$\widetilde{T}^\ell(v) = \{j \mid j \in T^\ell(v) \text{ and } e_j \text{ is successfully colored}\}.$$

Note that $|\widetilde{T}^\ell(v)| = |U^\ell(v)|$. Furthermore, recall that if e_i was not colored, then by definition $Z_i = Y_i(S) = X_i(S) = D_i(S) = p_i(S) = 0$ for all S.

Proposition 16. For any vertex v, subset of colors S, and phase r of v,

$$|\delta^r(v,S)| \leq \frac{1}{|A^r(v)|}\sum_{i\in T^{\leq r}(v)} Z_i + \left|\sum_{\ell=1}^{r}\frac{1}{|A^\ell(v)|}\sum_{i\in T^\ell(v)} D_i(S)\right| + \sum_{\ell=1}^{r}\frac{1}{|A^\ell(v)|}\sum_{i\in\widetilde{T}^\ell(v)}\left|p_i(S) - \frac{|A^{\ell-1}(v)\cap S|}{|A^{\ell-1}(v)|}\right|$$

Proof. From Equation (2.6),

$$\delta^\ell(v,S) - \delta^{\ell+1}(v,S) = \frac{1}{|A^{\ell+1}(v)|}\left(|U^{\ell+1}(v)\cap S| - |A^\ell(v)\cap S|\cdot\frac{|U^{\ell+1}(v)|}{|A^\ell(v)|}\right).$$

By definition $\delta^0(v,S) = 0$ and so:

$$\begin{aligned}
&|\delta^r(v,S)| \\
&= \left|\sum_{\ell=1}^{r}\delta^{\ell-1}(v,S) - \delta^\ell(v,S)\right| \\
&= \left|\sum_{\ell=1}^{r}\frac{1}{|A^\ell(v)|}\left(|U^\ell(v)\cap S| - |U^\ell(v)|\cdot\frac{|A^{\ell-1}(v)\cap S|}{|A^{\ell-1}(v)|}\right)\right| \\
&= \left|\sum_{\ell=1}^{r}\frac{1}{|A^\ell(v)|}\left(\sum_{i\in T^\ell(v)} Y_i(S) - |U^\ell(v)|\cdot\frac{|A^{\ell-1}(v)\cap S|}{|A^{\ell-1}(v)|}\right)\right| \\
&= \left|\sum_{\ell=1}^{r}\frac{1}{|A^\ell(v)|}\left(\sum_{i\in T^\ell(v)} (Y_i(S) - X_i(S)) + \sum_{i\in T^\ell(v)} X_i(S) - |U^\ell(v)|\cdot\frac{|A^{\ell-1}(v)\cap S|}{|A^{\ell-1}(v)|}\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{\ell=1}^{r} \frac{1}{|A^\ell(v)|} \sum_{i \in T^\ell(v)} |Y_i(S) - X_i(S)| + \left| \sum_{\ell=1}^{r} \frac{1}{|A^\ell(v)|} \left(\sum_{i \in T^\ell(v)} X_i(S) - |U^\ell(v)| \cdot \frac{|A^{\ell-1}(v) \cap S|}{|A^{\ell-1}(v)|} \right) \right| \\
&\leq \sum_{\ell=1}^{r} \frac{1}{|A^\ell(v)|} \sum_{i \in T^\ell(v)} Z_i \\
&\quad + \left| \sum_{\ell=1}^{r} \frac{1}{|A^\ell(v)|} \left(\sum_{i \in T^\ell(v)} (X_i(S) - p_i(S)) + \sum_{i \in T^\ell(v)} p_i(S) - |U^\ell(v)| \cdot \frac{|A^{\ell-1}(v) \cap S|}{|A^{\ell-1}(v)|} \right) \right| \\
&= \sum_{\ell=1}^{r} \frac{1}{|A^\ell(v)|} \sum_{i \in T^\ell(v)} Z_i + \left| \sum_{\ell=1}^{r} \frac{1}{|A^\ell(v)|} \left(\sum_{i \in T^\ell(v)} D_i(S) + \sum_{i \in \widetilde{E}_\ell(v)} \left(p_i(S) - \frac{|A^{\ell-1}(v) \cap S|}{|A^{\ell-1}(v)|} \right) \right) \right| \\
&\leq \frac{1}{|A^r(v)|} \sum_{i \in T^{\leq r}(v)} Z_i + \left| \sum_{\ell=1}^{r} \frac{1}{|A^\ell(v)|} \sum_{i \in T^\ell(v)} D_i(S) \right| + \sum_{\ell=1}^{r} \frac{1}{|A^\ell(v)|} \sum_{i \in \widetilde{E}_\ell(v)} \left| p_i(S) - \frac{|A^{\ell-1}(v) \cap S|}{|A^{\ell-1}(v)|} \right|
\end{aligned}
$$

$\square$

2.3.2 Technical parameters

Here we collect the parameters that are used in the next section. The margin of error for all of our bad events will be $\alpha\Delta$, where α is a constant depending only on ϵ and a constant M (specified below.) Given α, we will choose a value of b (the number of phases) that suits our purposes. We will also allow a constant number, C, of exceptions (see Definition 19), where C depends only on α. Finally, we will use these constants to define the constant N s.t. $n \leq 2^{\frac{\Delta}{N}}$ in the random order case.

Definition 17 (Technical parameters)**.**

- $\epsilon \in (0, 1)$ is the parameter appearing in the statements of Theorem 7 and Theorem 6.
- n always represents the number of vertices in the graph.
- $m = \lfloor n\Delta/2 \rfloor$ is the total number of time steps.

In the dense case, there is a *density parameter* $M > 1$ that is an upper bound on n/Δ. For notational convenience we will say that $M = 0$ in the random-order case.

There are several parameters given below that arise in the analysis. All of the parameters depend on ϵ and on M. As indicated above, the value $M = 0$ is used to refer to the random case.

- $\zeta = \zeta(\epsilon, M)$ is the scaling coefficient. In the random-order case, $\zeta(\epsilon, 0) = e^{-20/\epsilon^2}\frac{\epsilon^3}{10}$. In the dense case, for $M > 1$, $\zeta(\epsilon, M) = \frac{\epsilon^5}{100M}e^{-(5M/\epsilon^2)^2}$.

- $\alpha = \alpha(\epsilon, M) = \zeta\frac{\epsilon^3}{5}$.

- $b = b(\epsilon, M) = \frac{40}{\alpha\epsilon^2}$. This is the number of phases in the phase-partition for each vertex.

- $C = C(\epsilon, M) = \frac{2000}{\alpha^4}$. For each subset S of colors we say that a vertex v is S-atypical (Definition 19) if (very roughly) at some point in the algorithm the fraction of free colors at v that belong to S differs significantly from, $\frac{|S|}{(1+\epsilon)\Delta}$, the overall fraction of colors that belong to S. One of the bad events is that for some color set S, the number of S-atypical vertices is at least C.

The final parameter is only relevant for the random case:

- $N = N(\epsilon) = \max(400C(\epsilon, 0), 50b(\epsilon, 0))$. The theorem for the random case requires that $\Delta = \Omega(\log n)$. The parameter N is the lower bound on $\Delta/\log(n)$ for which the result holds.

2.3.3 Bad Events

In this section, we will define certain bad events for the run of the algorithm. These bad events are that some "error quantities" associated with the algorithm grow too large. These bad events, and their likelihood of occurring will be defined in terms of parameters in Definition 17. Note that our parameters vary depending on the random-order or dense case. In particular, in the dense case they depend on M.

We now identify three bad events, each associated with one of the three summands in Proposition 16. If none of them occur, we say that the resulting coloring is *well-behaved*. In this section we show that the coloring is very likely to be well-behaved. In the next section we show that in the two situations (an oblivious strategy that uses an arbitrary graph and random order, or an adaptively chosen dense graph) a well-behaved coloring will not have any uncolored edges.

The reader is reminded that various technical parameters are collected in Definition 17. The key parameter in this section is α.

The first type of bad event will occur if there are too many collisions at a particular vertex. This event corresponds directly to the first summand in Proposition 16.

Definition 18 (Too Many Collisions, $\mathcal{W}(v)$)**.** Given a vertex v, the bad event $\mathcal{W}(v)$ occurs if there exists a $j \in \{1, \ldots, m\}$ such that:

$$\sum_{i \in T(v), i \leq j} (Z_i - q_i) > \alpha\Delta.$$

The second type of bad event relates to the second summation in Proposition 16. As mentioned earlier, we can't hope to say that the summation is suitably small for all choices of S and v but it will be enough that for all S it is small for all but constantly many v.

Definition 19 (Too many S-atypical vertices, $\mathcal{B}(S)$)**.** For a vertex v and color set S, we say that v is S-*atypical* if there is a v-phase $1 \leq r \leq b$ such that:

$$\left| \sum_{\ell=1}^{r} \frac{1}{|A^{\ell}(v)|} \sum_{i \in T^{\ell}(v)} D_i(S) \right| > \frac{\alpha}{\epsilon}$$

Let $B(S)$ be the set of S-atypical vertices. We say that the bad event $\mathcal{B}(S)$ occurs if $|B(S)| \geq C$. (Here α and C are as given in Definition 17.)

The final family of bad events helps track $|F_i(u) \cap F_i(v)|$ for an edge $e = (u, v)$. This will ultimately be used to show that no edge runs out of colors.

Definition 20 (Too Much Drift at a pair of vertices, $\mathcal{D}(u, v)$)**.** Given a pair of vertices u, v, let $S_i = F_i(u) \cap F_i(v)$ be the set of colors free at both u and v at time i. Then, the bad event $\mathcal{D}(u, v)$ occurs if,

$$\left| \sum_{\substack{i \in T(u) \cup T(v) \\ j_1 \leq i \leq j_2}} D_i(S_i) \right| > \alpha \Delta$$

for any $1 \leq j_1 \leq j_2 \leq m$.

Next we will bound the probability of too many bad events occurring to show that the algorithm succeeds with high probability.

Definition 21 (Well-behaved Coloring)**.** We say that a **coloring is well-behaved** if:

(a) There are no vertices v such that $\mathcal{W}(v)$ occurs.

(b) There are no sets S such that $\mathcal{B}(S)$ occurs.

(c) There are no pairs of vertices u, v such that $\mathcal{D}(u, v)$ occurs.

Lemma 22. If $n \leq 2^{\frac{\Delta}{N}}$, then with probability at least $1 - \exp\left(-\frac{\alpha^4 \Delta}{1000}\right)$, the events Definition 21(a)-(c) do not occur, and consequently, the coloring is well-behaved.

We emphasize that this Lemma applies even for adaptive adversaries, and to sparse graphs, provided that $\Delta \geq N \log(n)$.

Proof. We show that the coloring is well-behaved by enumerating over each of the conditions (a)-(c), and bounding the probability they fail.

(a) Fix a vertex v and time $1 \le j \le m$. Apply the first part of Lemma 13 with

$$\beta_i = \begin{cases} 1 & i \in T(v), i \le j \\ 0 & \text{otherwise} \end{cases}.$$

Since the event that $i \in T(v)$ depends only on $\mathcal{F}_{i-1}$, the same holds for β_i. We have $|\beta_i| \le 1$ and $\sum |\beta_i| \le |T(v)| \le \Delta$, Applying the first part of Lemma 13 we obtain:

$$\Pr\left[\sum_{i \in T(v), i \le j} (Z_i - q_i) > \alpha\Delta\right] = \Pr\left[\sum_{i=1}^{m} \beta_i (Z_i - q_i) > \alpha\Delta\right] \le \exp\left(-\frac{\alpha^4 \Delta}{128}\right).$$

Taking a union bound over at most $m \le n\Delta \le \Delta \cdot 2^{\frac{\Delta}{N}}$ choices for j, we see that

$$\Pr\left[\mathcal{W}(v) \text{ occurs}\right] \le \Delta \cdot 2^{\frac{\Delta}{N}} \cdot \exp\left(-\frac{\alpha^4 \Delta}{128}\right)$$

for any vertex v. Then taking a union bound over at most $2^{\frac{\Delta}{N}}$ vertices, we get

$$\begin{aligned} \Pr\left[\exists v \text{ s.t. } \mathcal{W}(v) \text{ occurs}\right] &\le \Delta \cdot 2^{\frac{2\Delta}{N}} \cdot \exp\left(-\frac{\alpha^4 \Delta}{128}\right) \\ &\le \exp\left(-\frac{\alpha^4 \Delta}{128} + \frac{2\Delta}{N} + \ln \Delta\right) \\ &\le \exp\left(-\frac{\alpha^4 \Delta}{500}\right), \end{aligned}$$

for Δ sufficiently large, using $N \ge C = \frac{2000}{\alpha^4}$ in the last line.

(b) Fix a set S of colors and set $v_1, ..., v_C$ of C vertices. By definition if $v_1, ..., v_C$ are all S-atypical, then for each $k \in \{1, \dots, C\}$ there is a v_k-phase $r_k \in \{1, \dots, b\}$

such that:

$$\left| \sum_{\ell=1}^{r_k} \frac{1}{|A^\ell(v_k)|} \sum_{i \in T^\ell(v_k)} D_i(S) \right| > \frac{\alpha}{\epsilon}$$

We can think of this sum as having the form $\sum_{i \geq 1} \beta_i^k D_i(S)$ where

$$\beta_i^k = \begin{cases} \frac{1}{|A^{\phi_i(v_k)}(v_k)|} & \text{if } i \in T^{\leq r_k}(v_k) \\ 0 & \text{otherwise,} \end{cases}$$

and then we might hope to apply Lemma 13. However, the lemma requires that β_i^k be determined by F_{i-1} and that is not the case here because $|A^{\phi_i(v)}(v)|$ depends on the number of edges incident on v through the end of v-phase $\phi_i(v)$ (and whether they are colored or not) and this is not determined by F_{i-1}.

We address this by constructing a family of *fixed coefficient sequences* which is large enough that one of them agrees with the above coefficient sequence. Now for each choice of C fixed coefficient sequences (one for each vertex) we will apply Lemma 13, and then take a union bound over all such choices.

We note that all of the above coefficients are of the form $1/|A^\ell(v)|$ where $|A^\ell(v)|$ is an integer between $\epsilon\Delta$ and $(1+\epsilon)\Delta$. Thus if $v_1, ..., v_C$ are all S-atypical, then for each $k \in \{1, \ldots, C\}$ there is a v_k-phase $1 \leq r_k \leq b$ and for each ℓ between 1 and b there is an integer $s_k^\ell \in [\epsilon\Delta, (1+\epsilon)\Delta]$ such that:

$$\left| \sum_{\ell=1}^{r_k} \frac{\Delta\epsilon}{s_k^\ell} \sum_{i \in T^\ell(v_k)} D_i(S) \right| > \alpha\Delta$$

Consider a fixed choice of r_k and $s_k^\ell : 1 \leq \ell \leq b, k \in \{1, \ldots, C\}$.

For each $k \in \{1, \ldots, C\}$, define the coefficient sequence β^k by

$$\beta_i^k = \begin{cases} \frac{\epsilon\Delta}{s_k^\ell} & i \in T^\ell(v_k) \text{ with } \ell \leq r_k \\ 0 & \text{otherwise} \end{cases}.$$

As before, $\{i \in T^\ell(v_k)\}$ is determined by $\phi_i(v)$, which is determined by $\mathcal{F}_{i-1}$, so the same holds for β_i^k. Then, since for all ℓ, k, $\left|\frac{\epsilon\Delta}{s_k^\ell}\right| \leq 1$, for all i, $\sum_k |\beta_i^k| \leq |\{v_k : i \in T(v_k)\}| \leq 2$, and for all k, $\sum_i |\beta_i^k| \leq |T(v_k)| \leq \Delta$, taking $a = 2$ in Lemma 13 gives us

$$\Pr\left[\forall k \in \{1, \ldots, C\} \sum_i \beta_i^k D_i > \alpha\Delta\right] \leq \exp\left(-\frac{C\alpha^4\Delta}{256}\right).$$

This time we take a union bound over at most b^C choices of r_k for each vertex and at most $(\Delta+1)^{bC} \leq 2^{bC}\Delta^{bC}$ choices of $\{s_k^\ell\}$ to get

$$\Pr\left[\text{For all } k,\ v_k \text{ is } S\text{-atypical}\right] \leq b^C \cdot \Delta^{bC} \cdot 2^{bC} \cdot \exp\left(-\frac{C\alpha^4\Delta}{256}\right).$$

Taking another union bound over at most $\binom{n}{C} \leq 2^{\frac{C\Delta}{N}}$ sets of C vertices and $2^{(1+\epsilon)\Delta}$ sets S gives us

$$\begin{aligned}
&\Pr\left[\exists S, v_1, ..., v_C \text{ s.t. } v_k \text{ is } S\text{-atypical } \forall k\right] \leq 2^{(1+\epsilon)\Delta} \cdot 2^{\frac{C\Delta}{N}+bC} \cdot b^C \cdot \Delta^{bC} \cdot \exp\left(-\frac{C\alpha^4\Delta}{256}\right) \\
&\leq \exp\left(-\frac{C\alpha^4\Delta}{128} + 2\Delta + \frac{C\Delta}{N} + bC + bC\ln\Delta + C\ln b\right) \\
&\leq \exp\left(-\Delta\right),
\end{aligned}$$

for Δ sufficiently large, since $C = \frac{2000}{\alpha^4}$ and $N \geq 400C$.

(c) Fix $u, v \in V$ and $1 \leq j_1 \leq j_2 \leq m$. Define the sequence β by:

$$\beta_i = \begin{cases} 1 & i \in T(u) \cup T(v), j_1 \leq i \leq j_2 \\ 0 & \text{otherwise} \end{cases}.$$

Then, since the event $\{i \in T(u) \cup T(v)\}$ depends only on $\mathcal{F}_{i-1}$, $|\beta_i| \leq 1$, and $\sum |\beta_i| \leq |T(u) \cup T(v)| \leq 2\Delta$, the first part of Lemma 13 with $a = 1$ gives us:

$$\Pr\left[\left|\sum_{\substack{i \in T(u) \cup T(v) \\ j_1 \leq i \leq j_2}} D_i(S_i)\right| > \alpha\Delta\right] \leq \exp\left(-\frac{(\alpha/2)^4 \Delta}{128}\right).$$

Taking a union bound over at most $(n\Delta)^2 \leq n^4 \leq 2^{\frac{4\Delta}{N}}$ choices for j_1, j_2 gives us

$$\Pr\left[\mathcal{D}(u, v) \text{ occurs}\right] \leq 2^{\frac{4\Delta}{N}} \cdot \exp\left(-\frac{\alpha^4 \Delta}{512}\right).$$

Taking another union bound over at most $n^2 \leq 2^{\frac{2\Delta}{N}}$ vertex pairs gives us

$$\begin{aligned} \Pr\left[\exists u, v \text{ s.t. } \mathcal{D}(u, v) \text{ occurs}\right] &\leq 2^{\frac{6\Delta}{N}} \cdot \exp\left(-\frac{\alpha^4 \Delta}{512}\right) \\ &\leq \exp\left(-\frac{\alpha^4 \Delta}{512} + \frac{4\Delta}{N}\right) \\ &\leq \exp\left(-\frac{\alpha^2 \Delta}{800}\right) \end{aligned}$$

for Δ sufficiently large, where in the last line we used $N \geq 400C = \frac{800000}{\alpha^4}$.

Thus, for Δ sufficiently large, our total probability of a bad event occurring is at most

$$\exp\left(-\frac{\alpha^2 \Delta}{500}\right) + \exp(-\Delta) + \exp\left(-\frac{\alpha^4 \Delta}{800}\right) \leq \exp\left(-\frac{\alpha^4 \Delta}{1000}\right).$$

$\square$

2.4 Error Bounds

Our goal is to show that if the coloring is well-behaved, then no edge has gone uncolored. A key part of this proof is to upper bound $\delta^r(v,S) = \left|\frac{|A^r(v)\cap S|}{|A^r(v)|} - \frac{|S|}{(1+\epsilon)\Delta}\right|$ for any set S and any vertex v that is S-typical. Recall this is the error at the end of phase r of v. The upper bound will be expressed in terms of $\widehat{\epsilon}^r(v)$, which we will define below. For edge e incident on v, we will us $e - v$ to denote the other vertex incident on e.

Definition 23 (Error Bounds)**.** For each vertex-phase pair (v,r), $\widehat{\epsilon}^{\,r}(v)$ is defined inductively as follows:

$$\widehat{\epsilon}^{\,0}(v) = 0$$
$$\widehat{\epsilon}^{\,r}(v) = \zeta + \frac{5}{\Delta\epsilon^2} \sum_{\substack{j\in T^{\le r}(v)\\ u=e_j-v}} \widehat{\epsilon}^{\,\phi_j(u)-1}(u),$$

The above inductive definition is well-defined since for $e_j = (u,v), j \in T^{\le r}(v)$, $\mathbf{last}^{\phi_j(u)-1}(u) < j \le \mathbf{last}^r(v)$, so $\widehat{\epsilon}^{\,r}(v)$ depends only on $\widehat{\epsilon}^{\,s}(u)$ s.t. $(v,r) \prec (u,s)$.

Definition 24. A phase partition counter ϕ is *useful for the ordered graph* (G,σ) provided that:

(a) For each vertex v, and each phase $j \in [b]$, phase j of v has at most $2\Delta/b$ edges incident on v. In this case, we will say that ϕ is *balanced* with respect to (G,σ).

(b) For all vertices v phases r:
$$\widehat{\epsilon}^{\,r}(v) \le \frac{\epsilon^3}{10}.$$

The proof in Section 2.5 that algorithm $\mathcal{A}'$ succeeds with high probability relies on the existence of a phase partition function that is useful for (G,σ). In the rest of this section we show that such a partition exists (1) for any edge ordering σ provided that the graph is sufficiently dense, and (2) for a random edge ordering σ, with high probability. We start with:

Definition 25 (Valid Paths from v)**.** Let $\mathcal{P}^r(v)$ be the set of paths $(x_0, x_1, \cdots, x_t) = (e_{i_1}, e_{i_2}, \cdots, e_{i_t})$ such that, $x_0 = v$, $i_1 \in T^{\leq r}(v)$, and for all $1 \leq k < t$, edge $\phi_{i_{k+1}}(x_k) < \phi_{i_k}(x_k)$. That is, $e_{i_{k+1}}$ arrives in an earlier phase of x_i than edge e_{i_k}. We also include the empty path of length 0 in this set.

Proposition 26.

$$\widehat{\epsilon}^{\,r}(v) \leq \zeta \sum_{P \in \mathcal{P}^r(v)} \left(\frac{5}{\Delta\epsilon^2}\right)^{l(P)}$$

where $l(P)$ is the length of the path P.

Proof. Order the terms $\widehat{\epsilon}^{\,r}(v)$ according to Definition 9. We will prove this by strong induction on the error terms in this partial order. The base case is trivial since $\widehat{\epsilon}^{\,0}(v) = 0$ for all v. Now, note that for any $j \in T^{\leq r}(v)$, the term $\widehat{\epsilon}^{\,\phi_j(u)-1}(u)$ where $u = e_j - v$ comes before $\widehat{\epsilon}^{\,r}(v)$ according to the partial order in Definition 9. Furthermore, for $e_{i_0} = (v, u), i_0 \in T^{\leq r}(v)$ with $s = \phi_{i_0}(u)$ and given any valid path $(e_{i_1}, \cdots, e_{i_t}) \in \mathcal{P}^{s-1}(u)$ with $s > \phi_{i_1}(u)$, the path $(e_{i_0}, e_{i_1}, \cdots, e_{i_t}) \in \mathcal{P}^r(v)$. Thus, we have

$$\begin{aligned}
\widehat{\epsilon}^{\,r}(v) &= \zeta + \frac{5}{\Delta\epsilon^2} \sum_{\substack{j \in T^{\leq r}(v) \\ u = e_j - v}} \widehat{\epsilon}^{\,\phi_j(u)-1}(u) \\
&\leq \zeta + \zeta \cdot \frac{5}{\Delta\epsilon^2} \sum_{\substack{j \in T^{\leq r}(v) \\ u = e_j - v}} \sum_{P \in \mathcal{P}^{\phi_j(u)-1}(u)} \left(\frac{5}{\Delta\epsilon^2}\right)^{l(P)} \\
&\leq \zeta \sum_{P \in \mathcal{P}^r(v)} \left(\frac{5}{\Delta\epsilon^2}\right)^{l(P)}
\end{aligned}$$

where we include the path of length 0 starting at v. $\square$

Using this upper bound, we now finally describe the phase partition functions we use in our two different settings, and show that under these phase partition functions, the corresponding error bounds don't grow too quickly.The parameters m and b are as in Definition 17.

Definition 27 (Phase Partition Counters)**.** We use the following phase partition counter sequences:

(a) **Phase Partition Counters** $\phi^D = \left\{\phi_i^D\right\}_{i\in[m]}$ **for the dense case**: For each vertex v, $\phi_0^D(v) = 0$, and for $i \in [m]$

$$\phi_i^D(v) = \left\lceil \frac{|T(v) \cap \{1, \cdots, i\}| \cdot b}{\Delta} \right\rceil$$

Less formally, the counter is the number of edges so far incident to v times b/Δ rounded up to the nearest integer.

(b) **Phase Partition Counters** $\phi^R = \left\{\phi_i^R\right\}_{i\in[m]}$ **for the random-order case**: For every vertex v, $\phi_i^R(v) = \left\lceil \frac{i \cdot b}{m} \right\rceil$ for $i \in [m]$.

The definition of ϕ^D immediately gives that it is balanced with respect to (G, σ) for any graph G of maximum degree Δ and ordering σ. On the other hand, ϕ^R may not always be balanced with respect to (G, σ). However, under a uniformly random choice of σ, this will hold with high probability. In order to show this, we will need the following concentration bounds.

Lemma 28. [44, Theorem 2.10] Let X be a hypergeometric random variable with parameters m, d and k, where $\mu = \mathbb{E}[X] = \frac{kd}{m}$, then,

$$\Pr[X \geq \mu + t] \leq \exp\left(-\frac{t^2}{2(\mu + \frac{t}{3})}\right).$$

The following lemma shows that for a random ordering, ϕ^R is almost certainly balanced.

Lemma 29 (Probability that ϕ^R is unbalanced)**.** Let Δ be sufficiently large and suppose G is a graph on $n \leq 2^{\Delta/N}$ vertices, where N is given in Definition 17, and m edges, where some of the edges may be null edges. Then the fraction of orderings σ of edges of G such that ϕ^R is not balanced with respect to (G, σ) is at most $\exp\left(-\frac{\Delta}{20b}\right)$.

Proof. Consider σ chosen uniformly at random from all orderings. Let E^r be the (multi)-set of edges in phase r (which is the same for all v by the definition of ϕ^R). We say that E^r is a multi-set because many of the edges can be multi-edges. Let $E^r(v)$ be the set of edges in E^r that are incident to v. Our goal is to show that for all $v \in V$ and $r \in [b]$, $|E^r(v)| \leq \frac{2\Delta}{b}$ with high probability. E^r is a uniformly random subset of the edges of size $k_r \in \{\lfloor \frac{m}{b} \rfloor, \lceil \frac{m}{b} \rceil\}$ and therefore $|E^r(v)|$ is a hypergeometric random variable with expectation $\deg(v) \cdot \frac{k_r}{m} \leq \frac{\Delta}{b} \cdot \left(1 + \frac{b}{m}\right) \leq \frac{\Delta}{b} \cdot \frac{3}{2}$, for Δ sufficiently large. Thus, by Lemma 28, we have

$$\begin{aligned}\Pr\left[|E^r(v)| \geq \frac{2\Delta}{b}\right] &\leq \exp\left(-\frac{(\Delta/b)^2/4}{3(\Delta/b) + (\Delta/b)/3}\right)\\ &\leq \exp\left(-\frac{3\Delta}{40b}\right).\end{aligned}$$

Taking a union bound over all choices of $v \in V$ and $r \in [b]$ gives us that the probability that ϕ^R is not balanced is at most

$$b \cdot 2^{\frac{\Delta}{N}} \cdot \exp\left(-\frac{3\Delta}{40b}\right) \leq \exp\left(-\frac{3\Delta}{40b} + \frac{\Delta}{50b} + \log(b)\right) \leq \exp\left(-\frac{\Delta}{20b}\right)$$

by Definition 17 for Δ sufficiently large. $\square$

We now show that the given phase counter functions give us sufficiently slow growth for the error bounds in each of their respective contexts.

Proposition 30 (Bound on $\widehat{\epsilon}$: Dense Case)**.** If $n \leq M\Delta$ for some constant $M \geq 1$, then under the phase partition function ϕ^D, we have

$$\widehat{\epsilon}^r(v) \leq \frac{\epsilon^3}{10}$$

for all $v \in V$ and $1 \leq r \leq b$. Thus, for any graph G on $n \leq M\Delta$ vertices and any ordering σ of its edges, the phase partition counter ϕ^D is useful for (G, σ).

Proof. Consider any such graph G and ordering σ. We first show that there are at most $\frac{n^t}{(\lfloor t/2 \rfloor)!}$ valid paths of length t starting from v. Note that it is enough to show this for t even since if there are at most r valid paths of length t, then there are at most rn valid paths of length $t+1$. Let $P = (e_{i_1}, ..., e_{i_t})$ be any valid path where $v \in e_{i_1}$. Note that the edges in even positions $e_{i_2}, e_{i_4}, ..., e_t$ determine the path. Furthermore, these edges can only be placed in reverse arrival order for the path to be valid, so choosing the set of edges determines the path. Therefore, there can be at most $\binom{n^2}{\lfloor t/2 \rfloor} \leq \frac{n^t}{\lfloor t/2 \rfloor !}$ such paths. Thus, if $n \leq M\Delta$, then, by Proposition 26,

$$\begin{aligned}
\widehat{\epsilon}^r(v) &\leq \zeta \sum_{t=0}^{\infty} \left(\frac{5}{\Delta \epsilon^2}\right)^t \left(\frac{n^t}{(\lfloor t/2 \rfloor)!}\right) \\
&\leq \zeta \sum_{t'=0}^{\infty} \frac{2(5M/\epsilon^2)^{2t'+1}}{t'!} \\
&\leq \frac{10M}{\epsilon^2} \cdot \zeta \sum_{t'=0}^{\infty} \frac{((5M/\epsilon^2)^2)^{t'}}{t'!} \\
&\leq \frac{10M}{\epsilon^2} \cdot \zeta e^{(5M/\epsilon^2)^2} \leq \frac{\epsilon^3}{10}
\end{aligned}$$

by our choice of $\zeta = \zeta(\epsilon, M)$ from Definition 17. □

Proposition 31 (Bound on $\widehat{\epsilon}$: Random Order Setting)**.** If σ is an ordering such that the phase partition counter ϕ^R is balanced with respect to (G, σ), then for $\zeta = \zeta(\epsilon, 0)$, ϕ^R satisfies:

$$\widehat{\epsilon}^r(v) \leq \frac{\epsilon^3}{10}$$

for all $v \in V$ and $1 \leq r \leq b$. Thus, for any graph G on $n \leq 2^{\frac{\Delta}{N}}$ vertices and any such ordering σ, ϕ^R is useful for (G, σ).

Proof. Let G be a graph on $N \leq 2^{\frac{\Delta}{N}}$ vertices and σ be an ordering such that ϕ^R is balanced with respect to (G, σ). Recall that under ϕ^R, the phase counters $\phi_i^R(v)$ for all $v \in V$ are updated in lockstep. Consequently, for any edge $e_j = (u, v)$, $\phi_j^R(u) = \phi_j^R(v)$. As before, we bound $\widehat{\epsilon}^r(v)$ by bounding $|\mathcal{P}^r(v)|$ in Proposition 26. Consider a valid path $P \in \mathcal{P}^r(v)$, where $P = (e_{i_1}, \cdots, e_{i_t})$. Let $(x_0, x_1, \cdots, x_t)$ be the sequence of vertices such that $e_{i_j} = (x_{j-1}, x_j)$. Recall P has the property that $v = x_0$ and $e_{i_1} \in T^{\leq r}(v)$ and for all $1 \leq k < t$, we have, $\phi_{i_k}^R(x_{i_k}) > \phi_{i_{k+1}}^R(x_{i_k})$. In other words, each $e_{i_{k+1}}$ arrives in an earlier phase of x_k than e_{i_k}. Under ϕ^R the phase-partition for all vertices is the same so there is an associated unique phase, r_k associated to e_{i_k} and $r_1 > r_2 > \cdots > r_t$. So, we count the number of paths by first picking r_i's and then fixing the edges themselves. The number of ways of picking r_i's is at most $\binom{b}{t}$. Now we show how to inductively choose e_{i_k}'s. The number of ways of choosing e_{i_1} after one has fixed r_1, is $\frac{2\Delta}{b}$ (by the definition of balance property in **??**). Having fixed edge e_{i_j} and r_{j+1} the number of ways of picking $e_{i_{j+1}}$ is at most $\frac{2\Delta}{b}$. Thus, we have, by Proposition 26,

$$\begin{aligned}
\widehat{\epsilon}^r(v) &\leq \zeta \sum_{t=0}^{b} \left(\frac{5}{\Delta\epsilon^2}\right)^t \binom{b}{t} \left(\frac{2\Delta}{b}\right)^t \\
&\leq \zeta \sum_{t=0}^{b} \binom{b}{t} \left(\frac{10}{b\epsilon^2}\right)^t \\
&= \zeta \left(1 + \frac{10}{b\epsilon^2}\right)^b \\
&\leq \zeta e^{\left(\frac{10}{\epsilon^2}\right)} \\
&\leq \frac{\epsilon^3}{10}
\end{aligned}$$

where the last inequality follows from our choice of $\zeta = \zeta(\epsilon, 0)$ in Definition 17. $\square$

2.5 Main Lemma

We begin with a brief summary of what we've shown so far and what remains to be shown. For the dense case with a fixed adaptive adversary, we are given a constant M such that $n \le \frac{\Delta}{M}$ and define $\zeta = \zeta(\epsilon, M)$. We run the game $\mathcal{A}'$ using the phase-counter ϕ^D on n vertices. Since $n \le \frac{\Delta}{M} \le 2^{\Delta/N}$ (for Δ sufficiently large), Lemma 22 implies that the coloring produced by $\mathcal{A}'$ is well-behaved with high probability. Furthermore, by Proposition 30, ϕ^D is useful for the resulting graph G and ordering σ of edges.

Similarly, in the random case, we set $\zeta = \zeta(\epsilon, 0)$ and assume $n \le 2^{\frac{\Delta}{N}}$ (where $N = N(\epsilon)$.) By Lemma 29, for a graph G on n vertices and a uniformly chosen ordering σ of its edges, with high probability, ϕ^R is balanced with respect to (G, σ). Conditioned on ϕ^R being balanced for (G, σ), Proposition 31 guarantees us that ϕ^R is useful for (G, σ). Finally, in this case also Lemma 22 tells us that the coloring produced by $\mathcal{A}'$ is well-behaved with high probability.

In this final section, we will show that if the coloring produced by $\mathcal{A}'$ is well-behaved and ϕ is a useful phase partition counter for the final ordered graph (G, σ), then $\mathcal{A}'$ must have successfully produced a proper coloring of G. We do this by inductively showing that vertices are *good* according to the following definition. In this definition, and all following definitions in this section, we will assume that we are given (G, σ) with its corresponding parameter ζ and useful phase counter ϕ, as defined above.

Definition 32 (Good Vertices)**.** A vertex v is *good for set S during its r^{th} phase* if

$$|\delta^{r-1}(v, S)| \le \frac{\widehat{\epsilon}^{\,r-1}(v) \cdot \Delta}{|A^{r-1}(v)|}.$$

Note that in this definition we say v is good with respect to S during phase r rather than $r - 1$, because the palette for v used during phase r is $A^{r-1}(v)$.

Lemma 33 (Main Lemma)**.** Let phase partition counter ϕ be useful for the ordered graph (G, σ). If the coloring is well-behaved, then for all vertex phase pairs (v, r), for all color sets S, if v is an S-typical vertex, then v is good for S during its r^{th} phase.

We will prove this lemma by induction on the pairs (v, r) according to the order $\prec$ and by bounding each of the three summands in Proposition 16. The next two propositions relate these terms to the error terms.

Proposition 34. For any set $S \subseteq \Gamma$, and any i with $e_i = (u, v)$, $\phi_i(u) = s$, and $\phi_i(v) = r$, the preliminary color set $A_{i-1}(u) \cap A_{i-1}(v) = A^{s-1}(u) \cap A^{r-1}(v)$, satisfies

$$\left|A^{s-1}(u) \cap A^{r-1}(v)\right| \geq \frac{\epsilon^2 \Delta}{1+\epsilon} - \left|\delta^{s-1}(u, A^{r-1}(v))\right| |A^{s-1}(u)|.$$

Proof. By the definition of $\delta^{s-1}(u)$ in (2.3) and the fact that $|A^{r-1}(v)|$ is always at least $\epsilon\Delta$,

$$\begin{aligned} |A^{s-1}(u) \cap A^{r-1}(v)| &= \frac{|A^{s-1}(u)||A^{r-1}(v)|}{(1+\epsilon)\Delta} + \delta^{s-1}(u, A^{r-1}(v)) \cdot |A^{s-1}(u)| \\ &\geq \frac{\epsilon^2 \Delta}{1+\epsilon} - |\delta^{s-1}(u, A^{r-1}(v))||A^{s-1}(u)|, \end{aligned}$$

□

Note that this bounds the preliminary colors available to $e_i = (u, v)$ in terms of $\delta^{s-1}(u, S)$. Next we establish bounds on q_i and $p_i(S)$ if we know that u is good with respect to $A^{r-1}(v)$ and $A^{r-1}(v) \cap S$ during its phase s.

Proposition 35. Let phase partition counter ϕ be useful for the ordered graph (G, σ) and let $e_i = (u, v)$, $\phi_i(u) = s$, and $\phi_i(v) = r$. For any set S, if u is good for $A^{r-1}(v)$ during its s^{th} phase, then:

(a) The number of colors available to edge e_i is not too low, that is,

$$|A^{s-1}(u) \cap A^{r-1}(v)| \geq \frac{2\epsilon^2 \Delta}{5}$$

(b) The probability of a collision is low:

$$q_i \le \frac{\alpha}{4}$$

(c) If additionally, u is good for $A^{r-1}(v) \cap S$ during its s^{th} phase, the probability that the preliminary color chosen for edge e_i hits S is close to what we would expect if the color was chosen randomly from $A^{r-1}(v)$:

$$\left| p_i(S) - \frac{|A^{r-1}(v) \cap S|}{|A^{r-1}(v)|} \right| \le \frac{5\widehat{\epsilon}^{\,s-1}(u)}{\epsilon^2}.$$

Proof. For the first part:

$$\begin{aligned}
|A^{s-1}(u) \cap A^{r-1}(v)| &\overset{(i)}{\ge} \frac{\epsilon^2\Delta}{1+\epsilon} - |\delta^{s-1}(u, A^{r-1}(v))||A^{s-1}(u)| \\
&\overset{(ii)}{\ge} \frac{\epsilon^2\Delta}{1+\epsilon} - \widehat{\epsilon}^{\,s-1}(u)\Delta \\
&\overset{(iii)}{\ge} \frac{\epsilon^2\Delta}{1+\epsilon} - \frac{\epsilon^3\Delta}{10} \\
&\overset{(iv)}{\ge} \frac{2\epsilon^2\Delta}{5}
\end{aligned}$$

Here, (i) follows from Proposition 34, (ii) is using the fact that u is good for $A^{r-1}(v)$ during phase s, (iii) is by Definition 24 since ϕ is useful, and (iv) is because $\epsilon < 1$.

Now, recall from Definition 14 that

$$q_i := 1 - \frac{|F_{i-1}(u) \cap F_{i-1}(v)|}{|A_{i-1}(u) \cap A_{i-1}(v)|} = \frac{|(A_{i-1}(u) \cap A_{i-1}(v)) \setminus (F_{i-1}(u) \cap F_{i-1}(v))|}{|A_{i-1}(u) \cap A_{i-1}(v)|}.$$

Since by assumption ϕ is balanced, at most $\frac{4\Delta}{b}$ colors from $A_{i-1}(u) \cap A_{i-1}(v)$ could have been used by the time we color (u, v), which gives us:

$$q_i \le \frac{|(A_{i-1}(u) \setminus F_{i-1}(u)) \cup (A_{i-1}(v) \setminus F_{i-1}(v))|}{|A_{i-1}(u) \cap A_{i-1}(v)|} \le \frac{4\Delta}{b \cdot |A_{i-1}(u) \cap A_{i-1}(v)|}. \tag{2.14}$$

Using this and the definition of b from Definition 17, we have:

$$q_i \leq \frac{4\Delta}{b \cdot |A^{s-1}(u) \cap A^{r-1}(v)|} \leq \frac{10}{b\epsilon^2} \leq \frac{\alpha}{4},$$

For the last part, again by (2.3), for any color set T:

$$|A^{s-1}(u) \cap T| = \delta^{s-1}(u,T) \cdot |A^{s-1}(u)| + \frac{|A^{s-1}(u)||T|}{(1+\epsilon)\Delta}, \tag{2.15}$$

so

$$\begin{aligned} p_i(S) - \frac{|A^{r-1}(v) \cap S|}{|A^{r-1}(v)|} &= \frac{|A^{s-1}(u) \cap A^{r-1}(v) \cap S|}{|A^{s-1}(u) \cap A^{r-1}(v)|} - \frac{|A^{r-1}(v) \cap S|}{|A^{r-1}(v)|} \\ &= \frac{\overbrace{|A^{s-1}(u) \cap A^{r-1}(v) \cap S||A^{r-1}(v)|}^{(a)} - \overbrace{|A^{r-1}(v) \cap S||A^{s-1}(u) \cap A^{r-1}(v)|}^{(b)}}{|A^{s-1}(u) \cap A^{r-1}(v)||A^{r-1}(v)|} \end{aligned}$$

We first expand (a), by letting $T = A^{r-1}(v) \cap S$ in Equation (2.15),

$$|A^{s-1}(u) \cap A^{r-1}(v) \cap S||A^{r-1}(v)| = |A^{s-1}(u)| \left(\frac{|A^{r-1}(v) \cap S|}{(1+\epsilon)\Delta} + \delta^{s-1}(u, A^{r-1}(v) \cap S) \right) |A^{r-1}(v)|$$

Similarly, expanding (b) by letting $T = A^{r-1}(v)$ in Equation (2.15), we have,

$$|A^{r-1}(v) \cap S||A^{s-1}(u) \cap A^{r-1}(v)| = |A^{s-1}(u)| \left(\frac{|A^{r-1}(v)|}{(1+\epsilon)\Delta} + \delta^{s-1}(u, A^{r-1}(v)) \right) |A^{r-1}(v) \cap S|$$

Substituting these terms back, and taking absolute value, we can upper bound $\left| p_i(S) - \frac{|A^{r-1}(v) \cap S|}{|A^{r-1}(v)|} \right|$ as follows:

$$\begin{aligned} &= \left| \frac{|A^{s-1}(u)| \left(\delta^{s-1}(u, A^{r-1}(v) \cap S) \cdot |A^{r-1}(v)| - |A^{r-1}(v) \cap S| \cdot \delta^{s-1}(u, A^{r-1}(v))\right)}{|A^{s-1}(u) \cap A^{r-1}(v)||A^{r-1}(v)|} \right| \\ &\leq \frac{|A^{s-1}(u)| \left(|\delta^{s-1}(u, A^{r-1}(v) \cap S)| \cdot |A^{r-1}(v)| + |A^{r-1}(v) \cap S| \cdot |\delta^{s-1}(u, A^{r-1}(v))|\right)}{|A^{s-1}(u) \cap A^{r-1}(v)| \cdot |A^{r-1}(v)|} \\ &\leq \frac{|A^{s-1}(u)| \left(|\delta^{s-1}(u, A^{r-1}(v) \cap S)| + |\delta^{s-1}(u, A^{r-1}(v))|\right)}{|A^{s-1}(u) \cap A^{r-1}(v)|} \end{aligned}$$

$$\leq \frac{2\widehat{\epsilon}^{s-1}(u)\Delta}{2\epsilon^2\Delta/5} = \frac{5\widehat{\epsilon}^{s-1}(u)}{\epsilon^2},$$

where the final inequality uses the first part and the assumption that u is good for $A^{r-1}(v)$ and $A^{r-1}(v) \cap S$ during phase r. □

We now turn to the proof of the main lemma.

Proof of Main Lemma. Assume the coloring is well-behaved. We proceed by induction on vertex-phase pairs. First note that since $\delta^0(v, S) = 0$ by definition, we know that for all vertices v and sets S, v is good for S during its 1^{st} phase. Now, for any pair (v, r) with $1 \leq r < b$, we would like to show that for any set S such that v is S-typical, v is good for S during its $(r + 1)^{th}$ phase.

Since we proceed to by strong induction on the vertex-phase pair, we fix (v, r) and suppose that for any $(u, s) \prec (v, r)$ and any set S' such that u is S'-typical, u is good for S' during its $(s + 1)^{th}$ phase. Consider any S such that v is S-typical. Since we know the coloring is well-behaved, we know that for any phase $\ell \leq r$ of v, $|B(A^{\ell-1}(v))|, |B(A^{\ell-1}(v) \cap S)| \leq C$ (recall Definition 19). Let

$$T^{\ell}_B(v) = \{i \in T^{\ell}(v) : e_i - v \in B(A^{\ell-1}(v)) \cup B(A^{\ell-1}(v) \cap S)\},$$

so that $|T^{\ell}_B(v)| \leq 2C$. Then, for any $i \in T^{\ell}(v) \backslash T^{\ell}_B(v)$, with $u = e_i - v$ and $\phi_i(u) = s+1$, $\mathbf{last}^s(u) < i \leq \mathbf{last}^r(v)$, so $(u, s) \prec (v, r)$. Thus, by the inductive hypothesis, since u is both $A^{\ell-1}(v)$-typical and $(A^{\ell-1}(v) \cap S)$-typical, u must be good for both $A^{\ell-1}(v)$ and $A^{\ell-1}(v) \cap S$ during its $(s + 1)^{th}$ phase.

Recall that by Proposition 16, $|\delta^r(v, S)|$ is upper bound by

$$\frac{1}{|A^r(v)|} \sum_{i \in T^{\leq r}(v)} Z_i + \left| \sum_{\ell=1}^{r} \frac{1}{|A^{\ell}(v)|} \sum_{i \in T^{\ell}(v)} D_i(S) \right| + \sum_{\ell=1}^{r} \frac{1}{|A^{\ell}(v)|} \sum_{i \in \widetilde{T}^{\ell}(v)} \left| p_i(S) - \frac{|A^{\ell-1}(v) \cap S|}{|A^{\ell-1}(v)|} \right|.$$

Since the coloring is well-behaved, we know that $\mathcal{W}(v)$ does not occur, and therefore,

$$\sum_{i \in T^{\leq r}(v)} Z_i - q_i \leq \alpha\Delta.$$

By Proposition 35,

$$\begin{aligned}
\sum_{i \in T^{\leq r}(v)} Z_i &\leq \alpha\Delta + \sum_{\ell \leq r} \sum_{i \in T^{\ell}(v) \setminus T_B^{\ell}(v)} q_i + \sum_{\ell \leq r} \sum_{i \in T_B^{\ell}(v)} q_i \\
&\leq \alpha\Delta + \sum_{\ell \leq r} \sum_{i \in T^{\ell}(v) \setminus T_B^{\ell}(v)} \frac{\alpha}{4} + \sum_{\ell \leq r} \sum_{i \in T_B^{\ell}(v)} 1 \\
&\leq \alpha\Delta + \frac{\alpha\Delta}{4} + 2bC \\
&\leq 2\alpha\Delta. \qquad (2.16)
\end{aligned}$$

Furthermore, since v is S-typical, we are guaranteed that

$$\left| \sum_{\ell=1}^{r} \frac{1}{|A^{\ell}(v)|} \sum_{i \in T^{\ell}(v)} D_i(S) \right| \leq \frac{\alpha}{\epsilon} \leq \frac{\alpha\Delta(1+\epsilon)}{\epsilon \cdot |A^r(v)|}.$$

Finally, again by Proposition 35,

$$\begin{aligned}
&\sum_{\ell=1}^{r} \frac{1}{|A^{\ell}(v)|} \sum_{i \in \widetilde{T}^{\ell}(v)} \left| p_i(S) - \frac{|A^{\ell-1}(v) \cap S|}{|A^{\ell-1}(v)|} \right| \\
=&\sum_{\ell=1}^{r} \frac{1}{|A^{\ell}(v)|} \sum_{i \in \widetilde{T}^{\ell}(v) \setminus T_B^{\ell}(v)} \left| p_i(S) - \frac{|A^{\ell-1}(v) \cap S|}{|A^{\ell-1}(v)|} \right| + \sum_{\ell=1}^{r} \frac{1}{|A^{\ell}(v)|} \sum_{i \in \widetilde{T}^{\ell}(v) \cap T_B^{\ell}(v)} \left| p_i(S) - \frac{|A^{\ell-1}(v) \cap S|}{|A^{\ell-1}(v)|} \right| \\
=&\sum_{\ell=1}^{r} \frac{1}{|A^{\ell}(v)|} \sum_{\substack{i \in \widetilde{T}^{\ell}(v) \setminus T_B^{\ell}(v) \\ s=\phi(e_i - v)-1}} \frac{5\widehat{\epsilon}^{\,s}(e_i - v)}{\epsilon^2} + \sum_{\ell=1}^{r} \frac{1}{|A^{\ell}(v)|} \sum_{i \in \widetilde{T}^{\ell}(v) \cap T_B^{\ell}(v)} 1 \\
\leq&\frac{1}{|A^r(v)|} \left(2bC + \sum_{\substack{i \in T^{\leq r}(v) \\ s=\phi(e_i - v)-1}} \frac{5\widehat{\epsilon}^{\,s}(e_i - v)}{\epsilon^2} \right)
\end{aligned}$$

Combining this gives us

$$|\delta^r(v, S)| \leq \frac{1}{|A^r(v)|}\left(2\alpha\Delta + \frac{1+\epsilon}{\epsilon}\alpha\Delta + 2bC + \frac{5}{\epsilon^2}\sum_{\substack{i\in T^{\leq r}(v)\\ s=\phi(e_i-v)-1}} \widehat{\epsilon}^{\,s}(e_i - v)\right)$$

$$\leq \frac{\Delta}{|A^r(v)|}\left(\zeta + \frac{5}{\epsilon^2\Delta}\sum_{\substack{i\in T^{\leq r}(v)\\ s=\phi(e_i-v)-1}} \widehat{\epsilon}^{\,s}(e_i - v)\right) = \frac{\widehat{\epsilon}^{\,r}(v)\Delta}{|A^r(v)|},$$

where the second inequality holds because $2\alpha + \frac{1+\epsilon}{\epsilon}\alpha + \frac{2bC}{\Delta} \leq \frac{5}{\epsilon}\alpha \leq \zeta$, for Δ sufficiently large. $\square$

Finally, we use the main lemma to show that in a well-behaved coloring, no edge could have been left uncolored.

Corollary 36. Let phase partition counter ϕ be useful with respect to the ordered graph (G, σ). If the coloring produced by the $\mathcal{A}'$ is well-behaved, then for every edge $e = (u, v)$, at all times $i \in [m]$, we have $|F_{i-1}(u) \cap F_{i-1}(v)| \geq \frac{\epsilon^2\Delta}{2^{h+1}}$, where $h = \lceil 8/\epsilon^2 \rceil$. Thus, no edge is left uncolored.

Proof. Suppose for contradiction that there is an edge $e = (u, v)$ and time i such that $|F_i(u) \cap F_i(v)| < \frac{\epsilon^2\Delta}{2^{h+1}}$. Let $S_j = F_{j-1}(u) \cap F_{j-1}(v)$ and e_t be the edge adjacent to u or v that uses up the last color so that $|S_t| \geq \frac{\epsilon^2\Delta}{2^{h+1}}$ and $|S_{t+1}| < \frac{\epsilon^2\Delta}{2^{h+1}}$. This gives us $|S_1| \geq |S_2| \geq \cdots \geq |S_t| \geq \frac{\epsilon^2\Delta}{2^{h+1}}$ and $|S_{t+1}| < \frac{\epsilon^2\Delta}{2^{h+1}}$. Let $R = \{k \in T(u) \cup T(v) : k \leq t\}$ be the set of arrival times of edges adjacent to either u or v and define the partition $\{R_j\}$ of R to be $R_0 = \{k \in R : |S_k| \geq \frac{\epsilon^2\Delta}{2}\}$ and for $h \geq j \geq 1$,

$$R_j = \left\{k \in R : \frac{\epsilon^2\Delta}{2^{j+1}} \leq |S_k| < \frac{\epsilon^2\Delta}{2^j}\right\}$$

Note that $\{R_j\}_{j\geq 0}^h$ are disjoint intervals, and therefore, $|\bigcup_j R_j| \leq |T(u) \cup T(v)| \leq 2\Delta$. We will derive a contradiction by establishing lower bounds on the size of each

R_j, and show their sum exceeds 2Δ. Since $S_1 = (1+\epsilon)\Delta$, $|S_i|$ can drop below $\epsilon^2\Delta/2$ only after at least $(1+\epsilon-\epsilon^2/2)\Delta$ edges incident on u or v have arrived, so we have

$$|R_0| \geq (1+\epsilon-\epsilon^2/2)\Delta. \tag{2.17}$$

Claim 37. For each $h \geq j \geq 1$, $|R_j| \geq \frac{\epsilon^2\Delta}{8} - 7\alpha\Delta 2^{j-2}$.

Proof. Since the coloring is well-behaved, for each $1 \leq r \leq b$, $|B(A^r(u))|, |B(A^r(v))| \leq C$, so if we let

$$R^B = \left\{ k \in R : k \in \bigcup_{r=1}^{b} (B(A^r(u)) \cup B(A^r(v))) \right\},$$

we have $|R^B| \leq 2bC$. Furthermore, for any $i \in R \setminus R^B$, we know that $e_i = (u, w)$ or $e_i = (v, w)$ for some vertex w. Since w is $A^r(u)$-typical and $A^r(v)$-typical for all $1 \leq r \leq b$, by Lemma 33, we know that w is good for $A^r(u)$ and $A^r(v)$ for all $1 \leq r \leq b$, during all of its phases. Then, since $0 \leq Z_i, q_i \leq 1$ for all i and $\mathcal{W}(v), \mathcal{W}(u)$ didn't occur (because the coloring is well-behaved), we have by Proposition 35

$$\begin{aligned}
\sum_{i\in R} Z_i &\leq \sum_{i\in T(u), i\leq t} Z_i + \sum_{i\in T(v), i\leq t} Z_i \\
&\leq 2\alpha\Delta + \sum_{i\in T(u), i\leq t} q_i + \sum_{i\in T(v), i\leq t} q_i \quad \text{(From Definition 21(a) and Lemma 22)} \\
&\leq 2\alpha\Delta + 2\sum_{i\in R} q_i \qquad \text{(From definition of } R\text{)} \\
&\leq 2\alpha\Delta + 2\sum_{i\in R\setminus R^B} q_i + 2\sum_{i\in R^B} q_i \\
&\leq 2\alpha\Delta + 4\Delta \cdot \frac{\alpha}{4} + 4bC \qquad \text{(From Proposition 35(b))} \\
&\leq 4\alpha\Delta.
\end{aligned}$$

Note that since $Z_i \in [0,1]$ for all i, this means that for any $R' \subseteq R$, $\sum_{i\in R'} Z_i \leq 4\alpha\Delta$.

Similarly, since $\mathcal{D}(u,v)$ did not occur, and the sets R_j are intersections of $T(u) \cup T(v)$ with intervals we have for all j,

$$\begin{aligned}
\sum_{i\in R_j} X_i(S_i) \le \left|\sum_{i\in R_j} p_i(S_i)\right| + \left|\sum_{i\in R_j} D_i(S_i)\right| &\le \sum_{i\in R_j} p_i(S_i) + 2\alpha\Delta \\
&\le \sum_{i\in R_j\setminus R^B} p_i(S_i) + 2\alpha\Delta + 2bC \\
&\le \sum_{i\in R_j\setminus R^B} p_i(S_i) + 3\alpha\Delta
\end{aligned}$$

for all j. Finally, for $i \in R_j \setminus R^B$, suppose without loss of generality that $i \in T^r(v)$, with $x = e_i - v$ and $\phi_i(x) = s$. Then by Proposition 35(a),

$$p_i(S_i) = \frac{|S_i|}{|A^{r-1}(v) \cap A^{s-1}(x)|} < \frac{\epsilon^2\Delta/2^j}{2\epsilon^2\Delta/5} \le 5 \cdot 2^{-j-1} \le 2^{-j+2},$$

so

$$\sum_{i\in R_j} X_i(S_i) \le |R_j \setminus R_B| 2^{-j+2} + 3\alpha\Delta.$$

On the other hand, since the colors of at least $\frac{\epsilon^2\Delta}{2^{j+1}}$ edges e_i with $i \in R_j$ must hit S_i for $i \le h$, we have

$$\frac{\epsilon^2\Delta}{2^{j+1}} \le \sum_{i\in R_j} Y_i(S_i) \le \sum_{i\in R_j} X_i(S_i) + \sum_{i\in R_j} Z_i \le \sum_{i\in R_j} X_i(S_i) + 4\alpha\Delta.$$

Therefore,

$$\frac{\epsilon^2\Delta}{2^{j+1}} - 4\alpha\Delta \le \sum_{i\in R_j} X_i(S_i) \le |R_j \setminus R_B| \cdot 2^{-j+2} + 3\alpha\Delta$$

so

$$|R_j| \ge |R_j \setminus R_B| \ge \frac{\epsilon^2\Delta}{8} - 7\alpha\Delta \cdot 2^{j-2},$$

to complete the proof of the claim. $\square$

Summing the lower bounds on R_j for $1 \le j \le h$ yields:

$$\left|\bigcup_{j=1}^{h} R_j\right| \ge \Delta - 7\alpha\Delta \cdot 2^h \ge \left(1 - \frac{\epsilon^3}{10}\right)\Delta$$

since $7\alpha 2^{8/\epsilon^2+1} < 14\alpha e^{8/\epsilon^2} \le 14\zeta e^{8/\epsilon^2} \le \frac{\epsilon^3}{10}$ by our choice of ζ from Definition 17. Combining with (Equation (2.17)) and the fact that $\epsilon \le 1$, yields the desired contradiction:

$$\left|\bigcup_{j=0}^{h} R_j\right| > 2\Delta.$$

□

We can now complete the proofs of Theorem 7 and Theorem 6. According to Corollary 36, $\mathcal{A}'$ succeeds provided the partition function is useful with respect to (G, σ) and the coloring is well-behaved.

In the dense case of Theorem 7, Proposition 30 ensures that there is a partition function ϕ^D that is useful for (G, σ). Note that for any M, if Δ is sufficiently large, then $n \le M\Delta$ implies $n \le 2^{\frac{\Delta}{N}}$. Thus, in this case Lemma 22 implies that the coloring is well-behaved with probability at least $1 - 2^{-\alpha^4\Delta/1000}$, so this upper bounds the probability that $\mathcal{A}'$ (and also $\mathcal{A}$) fails to color (G, σ). Thus, letting $\gamma = \frac{\alpha^4}{1000}$, we have our proof for Theorem 7.

In the random case of Theorem 6, from Lemma 29, the phase counter ϕ^R is balanced with respect to (G, σ) with probability at least $1 - 2^{-\Delta/(20b)}$, and combined with Proposition 31 this ensures that the partition function is useful (G, σ). As in the dense case the probability that the coloring is not well behaved is at most $2^{-\alpha^4\Delta/1000}$. Therefore for all but at most a $2^{-\frac{\Delta}{20b}}$ fraction of edge orderings, the probability that $\mathcal{A}$ succeeds against $\mathbf{obl}(G, \sigma)$ is at least $1 - 2^{-\alpha^4\Delta/1000}$. Thus, letting $\gamma_1 = \frac{1}{20b}$ and $\gamma_2 = \frac{\alpha^4}{1000}$, we have our claim.

Proof of Corollary 3. Corollary 3 now follows from [13]. We give a brief sketch of the argument here. As mentioned above, the outcome of $\mathcal{A}$ against an adaptive adversary can be seen as a two player game between a Builder and Colorer. Note that this is a finite two player game with perfect information and no draws, so either Builder or Colorer must have a deterministic winning strategy. Our result above gives a random strategy for Colorer that ensures a win with high probability, which implies that Colorer must be the one with the deterministic winning strategy. This corresponds to a deterministic online coloring algorithm that succeeds on all graphs.

Chapter 3

Existence of Defending Distributions

This chapter is joint work with Michael Saks.

3.1 Introduction

Let A be a finite set of n elements and $w : A \to (0, 1)$ be an associated weight function. Furthermore, let

$$\mathcal{S} = \left\{ S \subseteq A \;\middle|\; \sum_{a \in S} w(a) \leq 1 \right\}$$

be the set of subsets of weight at most 1. We study probability distributions on $\mathcal{S}$. In particular, given any distribution μ on A, we would like to find a distribution ν on $\mathcal{S}$ s.t. for any total ordering $\succ$ on A,

$$\Pr_{a \sim \mu, S \sim \nu}[a \succ a' \;\forall\; a' \in S] < \mathop{\mathbb{E}}_{a \sim \mu} w(a). \tag{3.1}$$

We will call the distribution ν a *defending distribution* for μ. The following conjecture is (rephrased) from [45].

Conjecture 2. For any weight function w and any distribution μ on A, there exists a defending distribution ν for μ.

The focus of this chapter is on our attempts to make progress on this conjecture.

3.1.1 Background

Note: the technical results of this chapter do not depend on the contents of this section. A reader mainly interested in the former can skip to Section 3.1.2.

In this section we give a brief overview of the origin of this problem and its implications in the field of social choice theory. At its most general, social choice theory attempts to quantify concepts such as fairness, efficiency, and utility in the realm of societal decision making. Although it originates with problems regarding voting schemes and resource allocation (such as the cake cutting problem) it has since evolved to include a broader ranger of collective decision making problems, including multi-winner elections [34, 6, 52, 26] and participatory budgeting [35, 50]. A recent paper by Jiang, Munagala, and Wang ([45]) defines a committee selection problem that encapsulates both of the above types of problems, as well as many others. In their model, there is a set $\mathcal{N} = [n]$ of voters and a set $\mathcal{C} = [m]$ of candidates, and the goal is to select a committee of candidates in a manner that is fair to the voters. Each candidate i has a weight $s_i \geq 0$, and there is a limit K on the total weight of the selected committee (we will use $t(C)$ to denote the total weight of a committee C, which will be the sum of the weights of the candidates it contains.) Furthermore, each voter v has a preference order $\preceq_v$ over committees. Our only assumption on these preference orderings is that they are monotone: if committee $C \subseteq C'$, then for any voter v, $C \preceq_v C'$.

There are different notions of what a fair selection might mean in this scenario, and much work has been done on identifying desirable properties of and methods of selecting committees under different models for eliciting voter preferences, including approval ballots [47, 6, 51, 50, 26], candidate rankings [50, 34, 26], and types of utility functions [5, 52] (see [53] for a survey of such results in the area of participatory budgeting.) A line of work by Munagala, et. al. ([35, 26, 45]) aims to adapt the game theoretic notion of the core, first defined in [55], to the domain of public goods allocation. In [45], a committee is said to be in the core (or equivalently, to be stable) under the following condition:

Definition 1 (Stable Committee)**.** Given committees C, C', let

$$V(C, C') := \{v \in [n] \mid C \prec_v C'\}$$

denote the number of voters who strictly prefer C' to C. Then a committee C' blocks committee C if

$$V(C, C') \geq n \cdot \frac{t(C')}{K},$$

and C is stable (or in the core) iff there is no other committee C' that blocks it.

They justify this definition with a fair taxation argument [38]: each voter can be said to control $\frac{K}{n}$ of the budget, and a committee C is stable if no subset of voters are able to take their share of the overall budget to select a committee C' which they all strictly prefer and pay its cost, $t(C')$. Unfortunately, it is not hard to see that a stable committee does not always exist. Consider the following example, given in [26]:

Example 38. Let $n = m = 6 \cdot \ell$, $K = 3$, and $s_i = 1$ for all $i \in [m]$. The voters have the following preferences over candidates:

Voter	Candidate Preference Order
v_1	$c_1 \succ c_2 \succ c_3 \succ c_4 \succeq c_5 \succeq c_6$
v_2	$c_2 \succ c_3 \succ c_1 \succ c_4 \succeq c_5 \succeq c_6$
v_3	$c_3 \succ c_1 \succ c_2 \succ c_4 \succeq c_5 \succeq c_6$
v_4	$c_4 \succ c_5 \succ c_6 \succ c_1 \succeq c_2 \succeq c_3$
v_5	$c_5 \succ c_6 \succ c_4 \succ c_1 \succeq c_2 \succeq c_3$
v_6	$c_6 \succ c_4 \succ c_5 \succ c_1 \succeq c_2 \succeq c_3$

Finally, a voter (strictly) prefers committee C to C' if they (strictly) prefer their highest ranked candidate in C to their highest ranked candidate in C'. Note that in this scenario, a committee of weight at most K must exclude at least 2 candidates from either $\{1, 2, 3\}$ or $\{4, 5, 6\}$. In either case, the two voters who rank those two candidates most highly can take their share of the budget and choose a candidate they both strictly prefer. Thus, a stable committee cannot exist.

A standard method for circumventing such difficulties is to randomize [23, 5, 21]. We can select, instead of a single committee, a distribution of committees. Such a distribution will be called a *lottery*, and we will say a lottery is stable if it satisfies the following condition:

Definition 2 (Stable Lottery)**.** A distribution Δ over committees of weight at most K is a stable lottery if, for all committees C',

$$\mathop{\mathbb{E}}_{C \sim \Delta}[V(C, C')] < n \cdot \frac{t(C')}{K}.$$

It was shown in [26] that under certain restrictions on the voter preferences, a stable lottery always exists, but it is not known whether this holds in general. The following argument, given in [45], shows that an affirmative answer to Conjecture 2 implies the existence of a stable lottery for all possible weight assignments and voter preferences.

We begin by applying von Neumann's minimax theorem. By definition, a stable lottery exists iff

$$\min_{\Delta} \max_{C'} \mathop{\mathbb{E}}_{C \sim \Delta} \left[V(C, C') - n \cdot \frac{t(C')}{K} \right] < 0,$$

where Δ is taken from all distributions of committees of weight at most K and C' is taken from the set of all committees. Thus, by duality, it is enough to show that

$$\max_{\Delta'} \min_{\Delta} \mathop{\mathbb{E}}_{C \sim \Delta, C' \sim \Delta'} \left[V(C, C') - n \cdot \frac{t(C')}{K} \right] < 0,$$

where Δ is again taken from all distributions of committees of weight at most K and Δ' is taken from all distributions of committees. Note that we can trivially assume μ consists only of committees of weight at most K, since adding committees of greater weights only decreases the left hand side. Now, suppose Conjecture 2 is true. We will show that in this case, for any distribution Δ', there exists a distribution Δ such that

$$\mathop{\mathbb{E}}_{C \sim \Delta, C' \sim \Delta'} \left[V(C, C') - n \cdot \frac{t(C')}{K} \right] < 0.$$

Let A be the set of all committees of weight at most K, and let w be the normalized associated weight function so that $w(C) = \frac{t(C)}{K}$. Then a distribution Δ' on committees of weight at most K corresponds exactly to a distribution μ on A. Assuming Conjecture 2 is true, there exists a defending distribution, ν, for μ. Then we can get Δ from ν by taking, for each $S \sim \nu$, the union of committees in S, $C = \cup_{a \in S} a$, whose combined total normalized weight must be at most $t(C) \leq \sum_{a \in S} t(a) \leq K$ (by our definition of a defending distribution.) By monotonocity, for any voter v, $a' \sim \mu$, $S \sim \nu$, and $C = \cup_{a \in S} a$, if there is any $a \in S$ s.t. $a \succeq_v a'$, then it must be the case that $C \succeq_v a'$. Thus, since ν is a defending distribution for μ, we must have, for all voters v,

$$\Pr_{C'\sim\Delta', C\sim\Delta}[C' \succ_v C] \leq \Pr_{a'\sim\mu, S\sim v}[a' \succ_v a \;\forall\; a \in S] < \mathop{\mathbb{E}}_{a'\sim\mu} w(a') = \mathop{\mathbb{E}}_{C'\sim\Delta'} \frac{t(C')}{K}.$$

Summing over all voters v, we get

$$\mathop{\mathbb{E}}_{C'\sim\Delta', C\sim\Delta}[V(C, C')] < n \cdot \frac{\mathbb{E}_{C'\sim\Delta'} t(C')}{K}.$$

3.1.2 Our Results

The following lemma, rephrased from [45] gives some intuition for why a defending distribution might exist:

Lemma 39. Suppose the weight function w is supported on a single value (i.e., every element has the same weight.) Then, for any distribution μ on A, there exists a defending distribution ν for μ.

Proof. Let $k \in (0, 1)$ be the weight of every element of A. In this case, for any distribution μ on A, we have $\mathbb{E}_{a\sim\mu} w(a) = k$. We will construct the defending distribution ν as follows: let $b = \lfloor \frac{1}{k} \rfloor$, and let ν independently draw elements $a_1, ..., a_b \sim \mu$ and take S to be the set of all selected elements. Clearly $S \in \mathcal{S}$, since $\sum_{a\in S} w(a) = kb \leq 1$. However, by symmetry, for any total ordering $\succ$ on A,

$$\Pr_{a\sim\mu, S\sim\nu}[a \succ a' \;\forall\; a' \in S] = \Pr_{a\sim\mu, a_1,\ldots,a_b\sim\mu}[a \succ a_i,\ 1 \leq i \leq b] = \frac{1}{b+1} < k.$$

□

They also prove the following theorem, rephrased below:

Theorem 6. *Suppose w is supported on $\{\frac{1}{3}, \frac{2}{3}\}$. Then, for any distribution μ on A, there exists a defending distribution ν for μ.*

We extend this result to the following two theorems.

Theorem 7. *Suppose w is supported on any two values $\{k_1, k_2\}$. Then, for any distribution μ on A, there exists a defending distribution ν for μ.*

Theorem 8. *Suppose w is supported on any three values $\{\frac{1}{b_1}, \frac{1}{b_1 b_2}, \frac{1}{b_1 b_2 b_3}\}$, where $b_1, b_2, b_3 \in \mathbb{Z}_{\geq 2}$. Then, for any distribution μ on A, there exists a defending distribution ν for μ.*

In our efforts to generalize the above two results, we found instances where our methods no longer applied. However, this appeared to be more a limit of our methods than an inherent property of the problem. With more specific restrictions, we were able to show the existence of stable lotteries even in cases where our primary methods were insufficient. The result below summarizes this work.

Theorem 9. *Suppose w is supported on three weights $k_1 > k_2 > k_3$ s.t. $k_1 \in (\frac{1}{2}, 1)$, $k_2 \in (\frac{1}{3}, \frac{1}{2}]$, and $k_1 + k_2 > 1$. Then for any distribution μ on A, it is possible to find a defending distribution.*

Unfortunately, it is not clear how to extend this proof in order to generalize the two main theorems above. A more detailed analysis of this issue is given in Section 3.4

3.2 Weight Functions Supported on Two Distinct Weights

In this section we will prove Theorem 7. We begin by proposing a class of defending distributions against any distribution μ. We then analyze some properties of these distributions that generalize the symmetry utilized in the proof of Lemma 39. Finally, we use the desired properties to select a defending distribution from the proposed class that satisfies our requirements.

First, given the ground set A and weight function $w : A \to \{k_1, k_2\}$ with $k_1 > k_2 \in (0, 1)$, let $A_i = \{a \in A \mid w(a) = k_i\}$ for $i = 1, 2$. Given any distribution μ on A with $\mu(A_i) = p_i > 0$ for $i = 1, 2$, we can decompose it into the conditional distributions μ_i on A_i so that $\mu = p_1\mu_1 + p_2\mu_2$. Now, let $b = \lfloor \frac{1}{k_1} \rfloor$, and for $0 \leq r \leq b$, let $c_r = \lfloor \frac{1-rk_1}{k_2} \rfloor$ be the maximum number of elements from A_2 that can be contained in a set $S \in \mathcal{S}$ already containing r elements from A_1. From this, we can define distributions ν_r on $\mathcal{S}$, for $0 \leq r \leq b$, which independently draw r elements from A_1 and c_r elements from A_2 and take their union. Finally, for any distribution λ on $\{0, 1, ..., b\}$, we get a valid distribution $\nu = \sum \lambda_r \nu_r$ on $\mathcal{S}$, where $\lambda_r = \lambda(r)$. Our goal in this section will be to choose the distribution λ so that ν is a defending distribution for μ.

For $i \in \{1, 2\}$, any total ordering $\succ$, and $0 \leq r \leq b$ let

$$q_i(r) = \Pr_{a\sim\mu_i, S\sim\nu_r}[a \succ a' \ \forall\ a' \in S] = \Pr_{a\sim\mu_i, a_1,\ldots,a_r\sim\mu_1, a_{r+1},\ldots,a_{r+c_r}\sim\mu_2}[a \succ a_j \ 1 \leq j \leq r + c_r] \tag{3.2}$$

be the probability that an element a independently selected from μ_i is ranked strictly higher than all elements in a set S selected from ν_r. Using this notation, our goal is to find λ s.t. for all possible orderings $\succ$,

$$\sum_{r=0}^{b} \sum_{i=1}^{2} \lambda_r p_i q_i(r) < p_1k_1 + p_2k_2. \tag{3.3}$$

As before, symmetry guarantees us that

$$(b + 1)q_1(b) \leq 1 < (b + 1)k_1 \tag{3.4}$$

and

$$(c_0 + 1)q_2(0) \leq 1 < \left(\left\lfloor \frac{1}{k_2} \right\rfloor + 1 \right) k_2 = (c_0 + 1)k_2. \tag{3.5}$$

However, we have no such bounds on the other terms in our sum. For instance, if our ordering ranked all elements in A_1 above all elements in A_2, we would have $q_1(0) = 1$, but if the ordering was reversed, the best the previous symmetry argument could guarantee us would be $q_2(b) \leq \frac{1}{c_b+1}$, where $\frac{1}{c_b+1} \geq k_2$. However, we observe that it is not possible for both of these to occur at once. The following claim leverages this observation to extend the symmetry properties from Lemma 39 to all terms $q_i(r)$ in Equation (3.3).

Claim 1. For any total ordering $\succ$ and $1 \leq r \leq b$, we have:

$$rq_1(r-1) + (c_r+1)q_2(r) \leq 1 < rk_1 + (c_r+1)k_2. \tag{3.6}$$

Proof. Fix an ordering $\succ$ and let

$$f_1(x,y) = \Pr_{a_1,\ldots,a_x \sim \mu_1, a_{x+1},\ldots,a_{x+y} \sim \mu_2} [a_1 \succ a_i \; 1 < i \leq x+y],$$

$$f_2(x,y) = \Pr_{a_1,\ldots,a_x \sim \mu_1, a_{x+1},\ldots,a_{x+y} \sim \mu_2} [a_{x+1} \succ a_i \; 1 \leq i < x+y]$$

be the probabilities that when x elements are chosen from μ_1 and y elements are chosen from μ_2, the unique highest ranked element (if one exists) is the first element from μ_1 (repectively, μ_2.) Note that by symmetry, since at most one element can be the unique highest ranked element and all elements from the same distribution are equally likely to be highest ranked, we have

$$xf_1(x,y) + yf_2(x,y) \leq 1.$$

It is also clear by definition that

$$q_1(r) = f_1(r+1, c_r) \;\text{ and }\; q_2(r) = f_2(r, c_r+1).$$

Moreover, since increasing the total number of elements drawn only decreases the probability of any particular element being the highest ranked, we have, for any $x' \geq x, y' \geq y$,

$$f_i(x', y') \leq f_i(x, y).$$

Finally, we note that for any $1 \leq r \leq b$

$$c_{r-1} = \left\lfloor \frac{1-(r-1)k_1}{k_2} \right\rfloor \geq \left\lfloor \frac{1-rk_1}{k_2} + 1 \right\rfloor = c_r + 1,$$

since removing an element of weight k_1 from a subset allows you to add at least one element of weight k_2 without increasing the total weight past 1. Combining these observations, we see that

$$\begin{aligned} rq_1(r-1) + (c_r+1)q_2(r) &\leq rf_1(r, c_{r-1}) + (c_r+1)f_2(r, c_r+1) \\ &\leq rf_1(r, c_r+1) + (c_r+1)f_2(r, c_r+1) \\ &\leq 1. \end{aligned}$$

Finally, we note that by our choice of c_r,

$$rk_1 + (c_r+1)k_2 = rk_1 + \left(\left\lfloor \frac{1-rk_1}{k_2} \right\rfloor + 1 \right) k_2 > rk_1 + \frac{1-rk_1}{k_2} \cdot k_2 = 1.$$

$\square$

Given this claim, the proof of Theorem 7 follows fairly easily. We provide the proof in full detail below, along with additional explanation in order to lay the groundwork for the proof of Theorem 8.

We begin with a diagram that summarizes the claim above. Each square in the illustration represents a value of r and contains two nodes corresponding to the terms

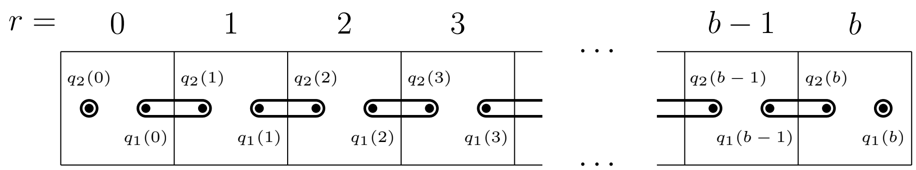

Figure 3.1: Relationships between failure probabilities of defending distributions ν_r in the case of two distinct weights.

$q_1(r), q_2(r)$ from the sum in Equation (3.3). For every inequality I of the form

$$L_I = m_1 q_1(r) + m_2 q_2(r') \leq 1 < m_1 k_1 + m_2 k_2 = R_I$$

given in Claim 1, Equation (3.4), or Equation (3.5), we say that $q_1(r), q_2(r') \in L_I$ if m_1, m_2, respectively are nonzero, and group these terms together in the diagram.

Note that every term is in at least one inequality. This suggests the following approach to a proof: let $\mathcal{I}$ be the set of inequalities shown above. We would like to find nonnegative weights $\{\beta_I \mid I \in \mathcal{I}\}$ and a distribution λ s.t.

$$\sum_{r=0}^{b} \sum_{i=1}^{2} \lambda_r p_i q_i(r) = \sum_{\mathcal{I}} \beta_I L_I, \tag{3.7}$$

where the two sides are seen as linear functions of the q_i terms that are algebraically equivalent. This would give us

$$\sum_{r=0}^{b} \sum_{i=1}^{2} \lambda_r p_i q_i(r) = \sum_{\mathcal{I}} \beta_I L_I < \sum_{\mathcal{I}} \beta_I R_I = \sum_{r=0}^{b} \sum_{i=1}^{2} \lambda_r p_i k_i = p_1 k_1 + p_2 k_2,$$

where the last part follows from the fact that the coefficient of $q_i(r)$ in any L_I is the coefficient of k_i in the corresponding R_I, and that $\sum_r \lambda_r = 1$. This will be our starting point in the proof below.

Proof of Theorem 7. As discussed above, given a weight function w supported on two values $k_1 < k_2$ and a distribution μ on A, in order to show that there is a distribution

ν of the proposed form which is a defending distribution, we need only show there exists a distribution λ s.t. Equation (3.7) holds. In this case, since every term appears in exactly one inequality, we only need to satisfy, for each inequality I of the form

$$rq_1(r-1) + (c_r+1)q_2(r) \leq 1 < rk_1 + (c_r+1)k_2$$

that

$$\beta_I L_I = \beta_I r q_1(r-1) + \beta_I(c_r+1)q_2(r) = \lambda_{r-1}p_1q_1(r-1) + \lambda_r p_2 q_2(r),$$

or equivalently that for all $1 \leq r \leq b$,

$$\frac{\lambda_{r-1}p_1}{r} = \beta_I = \frac{\lambda_r p_2}{c_r+1}.$$

In this setting, we can easily accomplish this by choosing λ s.t.

$$\lambda_r \propto \frac{p_1^r}{p_2^r \cdot r!} \cdot \prod_{j=1}^{r}(c_j+1).$$

$\square$

3.3 Weight Functions Supported on Three Distinct Divisible Weights

In this section we consider weight functions supported on 3 weights of the form $k_1 = \frac{1}{b_1}, k_2 = \frac{1}{b_1 b_2}$, and $k_3 = \frac{1}{b_1 b_2 b_3}$, where $b_1, b_2, b_3 \in \mathbb{Z}_{\geq 2}$. In this case, the sets in $\mathcal{S}$ have a nice structure that allows us to extend the symmetry arguments of the previous section to this setting. We begin by considering an analogue of Equation (3.3).

As before, we can let $A_i = \{a \in A \mid w(a) = k_i\}$ and, for any distribution μ on A, let $p_i = \mu(A_i)$ and μ_i be the conditional distribution of μ on A_i so that $\mu = \sum p_i \mu_i$. Then, let

$$\mathcal{R} = \{(r,t) \mid 0 \le r \le b_1,\ 0 \le t \le (b_1 - r)b_2\}.$$

For any $(r,t) \in \mathcal{R}$, we can let $c_{r,t} = ((b_1 - r)b_2 - t)b_3 = b_1b_2b_3 - rb_2b_3 - tb_3$ be the maximum number of elements from A_3 that can be contained in a set $S \in \mathcal{S}$ already containing r elements from A_1 and t elements from A_2. We can then once again define distributions $\nu_{r,t}$ on $\mathcal{S}$ that independently draw r elements from A_1, t elements from A_2, and $c_{r,t}$ elements from A_3 and take the union. Our family of possible defending distributions will now be parameterized by distributions λ on $\mathcal{R}$, where $\lambda_{r,t} = \lambda(r,t)$. Let $\nu = \sum_{(r,t)\in\mathcal{R}} \lambda_{r,t}\nu_{r,t}$ and for any ordering $\succ$, $(r,t) \in \mathcal{R}$, let

$$q_i(r,t) = \Pr_{a\sim\mu_i, S\sim\nu_{r,t}}[a \succ a'\ \forall\ a' \in S].$$

Then our goal in this case is to find λ s.t. for all possible orderings $\succ$,

$$\sum_{(r,t)\in\mathcal{R}} \sum_{i=1}^{3} \lambda_{r,t} p_i q_i(r,t) < \sum_{i=1}^{3} p_i k_i. \tag{3.8}$$

As in the two weight case, we have a set of inequalities between weighted sums of the terms $q_i(r,t)$ and sums of the terms k_i with identical coefficients. Before formally stating the full set of such inequalities, we first provide a diagram analogous to that of the two weight case that illustrates which terms belong in an inequality together for the special case that $b_1 = b_2 = b_3 = 2$. Different colors are used to denote inequalities with different forms.

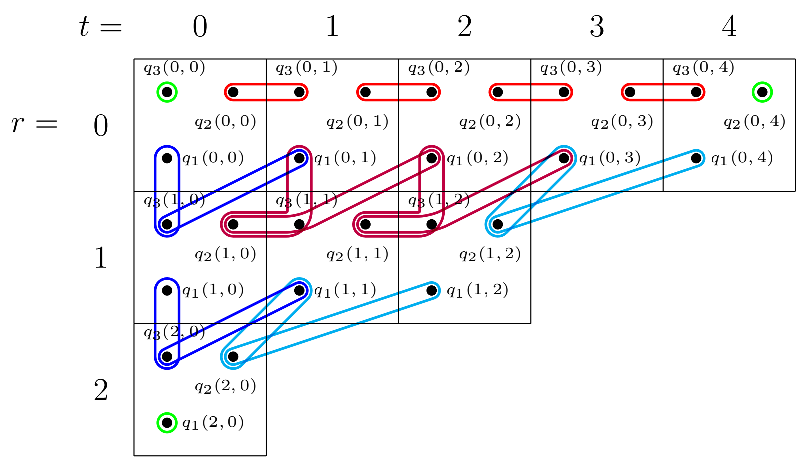

Figure 3.2: Relationships between failure probabilities of defending distributions ν_r in the case of three divisible weights.

We have the following analogue to Claim 1.

Claim 2. For any total ordering $\succ$, the following inequalities hold:

(a) Single-weight inequalities (circled in green above):

$$(b_1+1)q_1(b_1,0) \leq \ 1 \ < (b_1+1)k_1 \tag{3.9}$$

$$(b_1b_2+1)q_2(0,b_1b_2) \leq \ 1 \ < (b_1b_2+1)k_2 \tag{3.10}$$

$$(c_{0,0}+1)q_3(0,0) \leq \ 1 \ < (c_{0,0}+1)k_3 \tag{3.11}$$

(b) Furthermore, for any $0 < t \leq b_1b_2$ we have the following inequalities (circled in red above):

$$tq_2(0,t-1)+(c_{0,t}+1)q_3(0,t) \leq \ 1 \ < tk_2+(c_{0,t}+1)k_3 \tag{3.12}$$

(c) Similarly, for any $0 < r \leq b_1$, $0 \leq j < b_2$ we have the following inequalities (circled in dark blue above):

$$rq_1(r-1,j)+(c_{r,0}+1)q_3(r,0) \leq \ 1 \ < rk_1+(c_{r,0}+1)k_3 \tag{3.13}$$

(d) For any $0 < r \leq b_1$, $0 \leq j < b_2$ we have the following inequalities (circled in light blue above):

$$rq_1(r-1, b_1b_2 - rb_2 + 1 + j) + (b_1b_2 - rb_2 + 1)q_2(r, b_1b_2 - rb_2) \leq \ 1 \ < rk_1 + (b_1b_2 - rb_2 + 1)k_2 \tag{3.14}$$

(e) Finally, for any $0 < r \leq b_1$, $0 < t \leq b_1b_2 - rb_2$, and $0 \leq j < b_2$ we have the following inequalities (circled in purple above):

$$rq_1(r-1, t+j) + tq_2(r, t-1) + (c_{r,t} + 1)q_3(r,t) \leq \ 1 \ < rk_1 + tk_2 + (c_{r,t} + 1)k_3 \tag{3.15}$$

Proof. This can be proved similarly to Claim 1. First note that the second half of each of the above inequalities follows from simple computation and the definition of $c_{r,t}$, so we only need to show the first half for each type.

Now, fix an ordering $\succ$. For the sake of convenience, we introduce the following notation. For any pair of integers $(x, y) \notin \mathcal{R}$, let $q_i(x, y) = 0$ and $c_{x,y} = 0$. Then the claim for the first half of all of the above inequalities is encapsulated by the following statement: for any integers $x, y \geq 0$, and any integer $0 \leq j < b_2$,

$$xq_1(x-1, y+j) + yq_2(x, y-1) + (c_{x,y} + 1)q_3(x, y) \leq 1.$$

Note that, as before, we have

$$c_{x,y-1} = b_1b_2b_3 - xb_2b_3 - (y-1)b_3 \geq b_1b_2b_3 - xb_2b_3 - yb_3 + 1 = c_{x,y} + 1, \tag{3.16}$$

since removing an element of weight k_2 reduces the total weight of a subset by more than k_3, allowing you to add at least one such element without increasing the total weight past 1. Furthermore, we have

$$c_{x-1,y+b_2} = c_{x,y}, \tag{3.17}$$

since removing an element of weight k_1 changes the weight of the subset by the same amount as removing b_2 elements of weight k_2, so for $0 \leq j < b_2$,

$$c_{x-1,y+j} \geq c_{x-1,y+b_2-1} \geq c_{x-1,y+b_2} + 1 = c_{x,y} + 1. \tag{3.18}$$

For any $i \in \{1, 2, 3\}$ and integers $x_1, x_2, x_3 \geq 0$, let $f_j(x_1, x_2, x_3)$ be the probability that when we independently choose x_i elements from μ_i, the first element chosen from μ_j is the unique highest ranked out of them all. As in the previous case, by symmetry, we have

$$x_1 f_1(x_1, x_2, x_3) + x_2 f_2(x_1, x_2, x_3) + x_3 f_3(x_1, x_2, x_3) \leq 1,$$

and for $x' \geq x, y' \geq y, z' \geq z$,

$$f_i(x_1', x_2', x_3') \leq f_i(x_1, x_2, x_3).$$

Moreover, for any $x, y \geq 0$ and any integer j, we have by definition that

$$q_1(x-1, y+j) \leq f_1(x, y+j, c_{x-1,y+j}),$$
$$q_2(x, y-1) \leq f_2(x, y, c_{x,y-1}),$$
$$q_3(x, y) \leq f_3(x, y, c_{x,y} + 1).$$

Putting this all together, we see that for any integers $x, y \geq 0$, and any integer $0 \leq j < b_2$,

$$xq_1(x-1, y+j) + yq_2(x, y-1) + (c_{x,y} + 1)q_3(x, y)$$

$$\leq \; xf_1(x, y+j, c_{x-1,y+j}) + yf_2(x, y, c_{x,y-1}) + (c_{x,y}+1)f_3(x, y, c_{x,y}+1)$$

$$\leq \; xf_1(x, y+j, c_{x,y}+1) + yf_2(x, y, c_{x,y}+1) + (c_{x,y}+1)f_3(x, y, c_{x,y}+1) \quad \leq 1.$$

□

Let $\mathcal{I}$ be the set of inequalities given in Claim 2, and note that each $I \in \mathcal{I}$ has the form

$$L_I = m_1 q_1(r,t) + m_2 q_2(r,t) + m_3 q_3(r,t) \leq 1 < m_1 k_1 + m_2 k_2 + m_3 k_3 = R_I.$$

In order to prove Theorem 8, we would once again like to find weights $\{\beta_I\}$ and a distribution λ such that

$$\sum_{(r,t)\in\mathcal{R}} \sum_{i=1}^{3} \lambda_{r,t} p_i q_i(r,t) = \sum_{\mathcal{I}} \beta_I L_I. \tag{3.19}$$

as linear functions of the q_i terms so that

$$\sum_{(r,t)\in\mathcal{R}} \sum_{i=1}^{3} \lambda_{r,t} p_i q_i(r,t) = \sum_{\mathcal{I}} \beta_I L_I < \sum_{\mathcal{I}} \beta_I R_I = \sum_{(r,t)\in\mathcal{R}} \sum_{i=1}^{3} \lambda_{r,t} p_i k_i = \sum_{i=1}^{3} p_i k_i.$$

Unfortunately, we no longer have a unique inequality containing each term $q_i(r,t)$, so it is more difficult to solve for this set of weights. However, we can show that a solution exists. Before we prove Theorem 8, we first solve for the values of $\lambda_{r,t}$ that will make ν a valid defending distribution. It turns out that if Equation (3.19) is solvable, it determines a unique choice of λ.

Lemma 40. For any solution (λ, β) to Equation (3.19), λ must satisfy the following: for all $(r,t) \in \mathcal{R}$, $\lambda_{r,t} = \alpha_r \gamma_{r,t}$, where

$$\gamma_{r,t} = \frac{p_2^t}{p_3^t \cdot t!} \cdot \prod_{j=1}^{t} (c_{r,j}+1) \tag{3.20}$$

and the α_r's satisfy

$$\alpha_{r-1}\left(\sum_{(r-1,t)\in\mathcal{R}}\frac{\gamma_{r-1}p_1}{r}\right)=\alpha_r\left(\frac{\gamma_{r,b_1b_2-rb_2}p_2}{b_1b_2-rb_2+1}+\sum_{(r,t)\in\mathcal{R}}\frac{\gamma_{r,t}p_3}{c_{r.t}+1}\right). \tag{3.21}$$

These conditions uniquely determine the distribution λ.

Proof. We first solve for the relative values of $\lambda_{r,t}$ for a fixed r (across a row of the diagram.) Fix r and let $d=b_1b_2-rb_2$. Note that given r, for $0\le t<d$, any inequality I containing $q_2(r,t)$ also contains $q_3(r,t+1)$, and vice versa (these are exactly the inequalities given in Equation (3.12) or Equation (3.15), colored in red or purple in Figure 3.2.) Furthermore, in any such I, $q_2(r,t)$ always has coefficient $t+1$ and $q_3(r,t+1)$ always has coefficient $c_{r,t+1}+1$. Therefore, if we let $\mathcal{I}_{r,t}$ be the set of such I and sum the weighted coefficients of these terms across all $I\in\mathcal{I}_{r,t}$, Equation (3.19) gives us

$$\lambda_{r,t}p_2q_2(r,t)=\sum_{I\in\mathcal{I}_{r,t}}\beta_I(t+1)q_2(r,t)\quad\text{and}\quad\lambda_{r,t}p_3q_3(r,t)=\sum_{I\in\mathcal{I}_{r,t}}\beta_I(c_{r,t}+1)q_3(r,t)$$

which in turn gives us

$$\frac{\lambda_{r,t}p_2}{t+1}=\sum_{I\in\mathcal{I}_{r,t}}\beta_I=\frac{\lambda_{r,t+1}p_3}{c_{r,t+1}+1} \tag{3.22}$$

. This tells us that for a fixed r and $0\le t<d$, we must have $\lambda_{r,t+1}=\lambda_{r,t}\cdot\frac{p_2(c_{r,t+1}+1)}{p_3(t+1)}$. From this we conclude that the coefficients $\{\lambda_{r,t}\}$ satisfy that for $t>0$, $\lambda_{r,t}=\gamma_{r,t}\lambda_{r,0}$ and we can let $\alpha_r=\gamma_{r,t}$.

Next we need to determine the relative values of the α_r's. This time, for $0<r\le b$, we again let $d=b_1b_2-rb_2$ and let $\mathcal{I}_r$ be the set of inequalities containing terms of the form $q_1(r-1,t)$ for $0\le t\le d$, given by Equation (3.13), Equation (3.14), and Equation (3.15) (colored in dark blue, purple, and light blue, in Figure 3.2.) Note

that these are exactly the inequalities which contain terms of the form $q_2(r,t)$ or $q_3(r,t)$. Let $\mathcal{I}_r^3 \subseteq \mathcal{I}_r$ be the subset of inequalities which contain terms of the form $q_3(r,t)$ (given by Equation (3.13) and Equation (3.15), in dark blue or purple), and $\mathcal{I}_r^2 = \mathcal{I}_r \setminus \mathcal{I}_r^3$ be the remaining inequalities (which are all of the inequalities containing the term $q_2(r,d)$, given by Equation (3.14), in light blue.) For a fixed r, we must have

$$\sum_{(r-1,t)\in\mathcal{R}} \frac{\lambda_{r-1,t}p_1}{r} = \sum_{I\in\mathcal{I}_r} \beta_I = \sum_{I\in\mathcal{I}_r^2} \beta_I + \sum_{I\in\mathcal{I}_r^3} \beta_I = \frac{\lambda_{r,d}p_2}{d+1} + \sum_{(r,t)\in\mathcal{R}} \frac{\lambda_{r,t}p_3}{c_{r,t}+1} \tag{3.23}$$

which implies Equation (3.21). This determines α_r relative to α_{r-1} and thus determines $\lambda_{r,t}$ up to the choice of α_0. Finally, α_0 must be chosen so that the $\{\lambda_{r,t}\}$ sum to 1, so there is a unique choice. □

Proof of Theorem 8. Finally, given the above values of $\lambda_{r,t}$, we need to show that we can find corresponding nonnegative weights β_I for each of the inequalities. In other words, we need to find weights $\{\beta_I\}$ satisfying, for all r,t,

$$\frac{\lambda_{r,t}p_1}{r+1} = \sum_{I\ni q_1(r,t)} \beta_I, \tag{3.24}$$

$$\frac{\lambda_{r,t}p_2}{t+1} = \sum_{I\ni q_2(r,t)} \beta_I, \tag{3.25}$$

and

$$\frac{\lambda_{r,t}p_3}{c_{r,t}+1} = \sum_{I\ni q_3(r,t)} \beta_I, \tag{3.26}$$

so that Equation (3.19) holds. Note that if there is only one term on the right hand side of these equations for the inequalities given by Equation (3.9), Equation (3.10), Equation (3.11), and Equation (3.12) (colored in green and red in Figure 3.2) since the terms they contain are not contained by any other inequalities. In the case of the first three (green), this trivially determines the values of β_I for such I, since they each correspond to one unique term and their weights are determined by the weight of that term in the left hand side of the above equations. The weights on inequalities given in Equation (3.12) (red) depend on two terms and thus each correspond to two equations above, but by our choice of $\lambda_{r,t}$ (in particular Equation (3.22)), the weights induced by both are consistent. It only remains to determine weight for inequalities in $\mathcal{I}_r$ for $0 < r \leq b$ (given by Equation (3.13), Equation (3.14), Equation (3.15) and colored in dark blue, light blue, and purple, respectively.)

Fix r s.t. $0 < r \leq b$, $d = b_1 b_2 - r b_1$ with the aim of solving for the weights of the inequalities in $\mathcal{I}_r$. We first observe that each of these inequalities contains a term $q_1(r-1,t)$ for $0 \leq t \leq d + b_2$ in row $r-1$ and contains exactly one of $q_3(r,0)$ (dark blue), $q_2(r,d)$ (light blue), or the pair $q_2(r,t-1), q_3(r,t)$ (purple) for $1 \leq t \leq d$ in row r. For those inequalities containing a pair $q_2(r,t-1), q_3(r,t)$, we again have by Equation (3.22) that the corresponding equations of the form 3.25 and 3.26 are the same (since the left hand sides are equal, and the right hand sides are summing over the same sets of inequalities.) Thus, each inequality in $\mathcal{I}_r$ belongs to only two distinct equations, one corresponding to terms in row $r-1$, and one to terms in row r.

We can think of this as a flow problem, where each distinct equation corresponds to a vertex with capacity equivalent to the left hand side of that equation and each β_I indicates some amount of flow going through an edge between the vertices corresponding to the equations involving I. More precisely, we can construct a bipartite graph G_r with a source x, a sink y, and two sets V, U corresponding to terms in row $r-1$ and r, respectively. On one side, adjacent to x, we will have vertices v_t in V representing the terms $q_1(r-1,t)$ for $0 \leq t \leq d + b_2$, each with capacity

$$g(v_t) = \frac{\lambda_{r-1,t} p_1}{r}.$$

On the other side, adjacent to y, we have vertices u_t representing the pair of terms $q_3(r,t)$ and $q_2(r,t-1)$ for $1 \leq t \leq d$, each with capacity

$$g(u_t) = \frac{\lambda_{r,t} p_3}{c_{r,t}+1} = \frac{\lambda_{r,t-1} p_2}{t}.$$

Finally, we will add a vertex u_{d+1} adjacent to y representing the term $q_2(r,d)$, with capacity

$$g(u_{d+1}) = \frac{\lambda_{r,d} p_2}{d+1}$$

and a vertex u_0 adjacent to y representing the term $q_3(r,0)$, with capacity

$$g(u_0) = \frac{\lambda_{r,0} p_3}{c_{r,0}+1}.$$

Given these vertices, we will draw an edge for any inequality in $\mathcal{I}_r$ from the vertex v_i corresponding to its q_1 term to the vertex u_j corresponding to its q_2 and q_3 terms. Thus, we have an edge to each u_j from $v_j, v_{j+1}, ..., v_{j+b_2-1}$. Looking at Figure 3.2 and considering the blue and purple inequalities between rows 0 and 1, we can see the structure of the corresponding graph G_r for the case that $b_1 = b_2 = b_3 = 2$ and $r = 1$. We explicitly draw this graph G_r below:

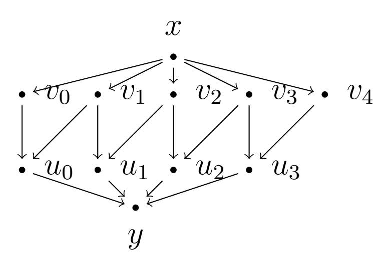

Figure 3.3: Example of a graph G modelling the relationships between terms in Equation (3.19)

Note that $\sum_{t=0}^{d+b_2} g(v_i) = \sum_{t=0}^{d+1} g(u_i)$ by our choice of $\lambda_{r,t}$ (see Equation (3.23).)

Thus, if we let

$$\kappa_r := \sum_{t=0}^{d+b_2} g(v_i) = \sum_{t=0}^{d+1} g(u_i)$$

and show that there is a flow of value κ in this graph (filling all vertices to their capacity), we can take the value of the flow on each edge (v_j, u_i) to be the weight of the corresponding inequality, and we will have weights that satisfy equations 3.24, 3.25, and 3.26. The following lemma completes our argument.

Lemma 41. There exists a flow of value κ_r from x to y in G_r.

□

In order to prove the lemma, we will use certain properties of the capacities of the vertices of G_r, summarized in the claim below.

Claim 3. In graph G_r, for any $0 \le t < d$, we have

$$\frac{g(v_{t+1})}{g(v_t)} \ge \frac{g(u_{t+1})}{g(u_t)}.$$

Furthermore, for any $b_2 - 1 \le t < d + b_2$, we have

$$\frac{g(v_{t+1})}{g(v_t)} \le \frac{g(u_{t-b_2+2})}{g(u_{t-b_2})}.$$

Proof of Claim 3. For any $0 \le t \le d$, we have

$$\frac{g(v_{t+1})}{g(v_t)} = \frac{\lambda_{r-1,t+1}}{\lambda_{r-1,t}} = \frac{\gamma_{r-1,t+1}}{\gamma_{r-1,t}} = \frac{p_2(c_{r-1,t+1}+1)}{p_3(t+1)}.$$

For $0 \le t < d$,

$$\frac{g(u_{t+1})}{g(u_t)} = \frac{\lambda_{r,t+1}(c_{r,t}+1)}{\lambda_{r,t}(c_{r,t+1}+1)} = \frac{\gamma_{r,t+1}}{\gamma_{r,t}} \cdot \frac{c_{r,t}+1}{c_{r,t+1}+1} = \frac{p_2(c_{r,t+1}+1)}{p_3(t+1)} \cdot \frac{c_{r,t}+1}{c_{r,t+1}+1} = \frac{p_2(c_{r,t}+1)}{p_3(t+1)} \le \frac{g(v_{t+1})}{g(v_t)},$$

where the last line follows from the fact that $c_{r,t} \le c_{r-1,t+1} + 1$ by Equation (3.18). If

$t = d$, then

$$\frac{g(u_{t+1})}{g(u_t)} = \frac{\lambda_{r,d} p_2(c_{r,d}+1)}{\lambda_{r,d} p_3(d+1)} = \frac{\gamma_{r,d} p_2(c_{r,d}+1)}{\gamma_{r,d} p_3(d+1)} = \frac{p_2(c_{r,d}+1)}{p_3(d+1)} \le \frac{g(v_{d+1})}{g(v_d)},$$

again since $c_{r,d} \le c_{r-1,d+1}$ by Equation (3.18).

On the other hand, if $b_2 - 1 \le t < d + b_2 - 1$, then

$$\begin{aligned}\frac{g(u_{t-b_2+2})}{g(u_{t-b_2+1})} &= \frac{\lambda_{r,t-b_2+2}(c_{r,t-b_2+1}+1)}{\lambda_{r,t-b_2+1}(c_{r,t-b_2+2}+1)} = \frac{\gamma_{r,t-b_2+2}}{\gamma_{r,t-b_2+1}} \cdot \frac{c_{r,t-b_2+1}+1}{c_{r,t-b_2+2}+1} \\ &= \frac{p_2(c_{r.t-b_2+2}+1)}{p_3(t-b_2+1)} \cdot \frac{c_{r,t-b_2+1}+1}{c_{r,t-b_2+2}+1} = \frac{p_2(c_{r,t-b_2+1}+1)}{p_3(t-b_2+1)} \ge \frac{p_2(c_{r-1,t+1}+1)}{p_3(t+1)} = \frac{g(v_{t+1})}{g(v_t)},\end{aligned}$$

since $t - b_2 + 1 \le t + 1$ and $c_{r,t-b_2+1} = c_{r-1,t+1}$ by Equation (3.17).

Finally, if $t = d + b_2 - 1$, then

$$\frac{g(u_{t-b_2+2})}{g(u_{t-b_2})} = \frac{\lambda_{r,d} p_2(c_{r,d}+1)}{\lambda_{r,d} p_3(d+1)} = \frac{p_2(c_{r,d}+1)}{p_3(d+1)} \ge \frac{p_2(c_{r-1,d+b_2}+1)}{p_3(d+b_2)} = \frac{g(v_{t+1})}{g(v_t)},$$

by the same reasoning as above. □

Proof of Lemma 41. We will use the Max-Flow-Min-Cut theorem [1] to show there is a flow of value κ for G, by showing that any xy vertex cut has at least this value. Consider any such cut S on G. Let $V' = V \setminus S$. Then we must have the non-x neighbors $N(V') \setminus \{x\} = U' \subseteq U$ all contained in S. Thus, the total weight of the cut is at least $\sum_{v \in V \cap S} g(v) + \sum_{u \in U'} g(u)$. We will show that $\sum_{u \in U'} g(u) \ge \sum_{v \in V'} g(v)$, which proves the result, since it implies

$$\sum_{v \in V \cap S} g(v) + \sum_{u \in U \cap S} g(u) \ge \sum_{v \in V \cap S} g(v) + \sum_{u \in U'} g(u) \ge \sum_{v \in V \cap S} g(v) + \sum_{v \in V'} g(v) = \sum_{v \in V} g(v) = \kappa.$$

We do this by proving the equivalent statement $\kappa \sum_{u \in U'} g(u) \ge \kappa \sum_{v \in V'} g(v)$ as follows. First, we'll define a function $\phi : V' \times U \to V \times U'$. We'll then show

[1] See [22] for a reference.

that ϕ is injective and furthermore, for any $(v', u) \in V' \times U$ with $\phi(v', u) = (v, u')$, $g(v')g(u) \leq g(v)g(u')$. This will allow us to conclude that

$$\kappa \sum_{v' \in V'} g(v') = \sum_{u \in U, v' \in V'} g(u)g(v') \leq \sum_{u \in U, v' \in V, (v,u') = \phi(v',u)} g(u')g(v) \leq \sum_{u' \in U', v \in V} g(u')g(v) = \kappa \sum_{u' \in U'} g(u').$$

We give the definition of ϕ below:

$$\phi(v_i, u_j) := \begin{cases} (v_i, u_j) & u_j \in U' \\ (v_j, u_i) & u_j \notin U', j > i \\ (v_{b_2-1+j}, u_{i-b_2+1}) & u_j \notin U', j \leq i - b_2 \end{cases}.$$

Note that if v_i is in V', then for any $i - b_2 + 1 \leq j \leq i$, $u_j \in U'$, so this function is well defined. This also implies that in the latter two cases, since $u_j \notin U'$, $v_j, v_{j+b_1-1} \notin V'$, so ϕ maps $V' \times U'$ to itself (acting as the identity in this case) and $V' \times (U \setminus U')$ to $(V \setminus V') \times U'$.

Now suppose there was some $(i, j) \neq (k, \ell)$ s.t. $\phi(v_i, u_j) = \phi(v_k, u_\ell)$. Clearly in this case we could not have u_ℓ or $u_j \in U'$, since that would mean $(v_i, u_j), \phi(v_i, u_j) = \phi(v_k, u_\ell), (v_k, u_\ell)$ would all have to be in $V' \times U'$ by the discussion above, and therefore would all be equal. We also can't have both $j > i$ and $\ell > k$, since that would give us $(v_j, u_i) = \phi(v_i, u_j) = \phi(v_k, u_\ell) = (v_\ell, u_k)$, or both $j \leq i - b_2, \ell \leq k - b_2$, since that would give us $(v_{b_2-1+j}, u_{i-b_2+1}) = \phi(v_i, u_j) = \phi(v_k, u_\ell) = (v_{b_2-1+\ell}, u_{k-b_2+1})$. Therefore it would have to be the case that $u_j, u_\ell \notin U'$, and WLOG $j > i$ and $\ell \leq k - b_2$. Then $i = k - b_2 + 1 < j$ and $j = b_2 - 1 + \ell \leq k - 1$. But this would mean that $u_j \in N(v_k)$ and $v_k \in V'$, so $u_j \in U'$, a contradiction. Thus, ϕ must be injective.

Finally, we verify that if $\phi(v', u) = (v, u')$, then $g(v')g(u) \leq g(v)g(u')$ in all three cases. If $u_j \in U'$ we obviously get equality. If $j > i$, then this is equivalent to showing that $\frac{g(v_j)}{g(v_i)} \geq \frac{g(u_j)}{g(u_i)}$, which is given to us by Claim 3, since

$$\frac{g(v_j)}{g(v_i)} = \frac{g(v_{i+1})}{g(v_i)} \cdot \ldots \cdot \frac{g(v_j)}{g(v_{j-1})} \leq \frac{g(u_{i+1})}{g(u_i)} \cdot \ldots \cdot \frac{g(u_j)}{g(u_{j-1})} = \frac{g(u_j)}{g(u_i)}.$$

On the other hand, if $j \leq i - b_2$, then we'd like to show that $\frac{g(v_i)}{g(v_{b_2-1+j})} \leq \frac{g(u_{i-b_2+1})}{g(u_j)}$, which again follows from Claim 3, since

$$\frac{g(v_i)}{g(v_{b_2-1+j})} = \frac{g(v_{b_2+j})}{g(v_{b_2-1+j})} \cdot \ldots \cdot \frac{g(v_i)}{g(v_{i-1})} \leq \frac{g(u_{j+1})}{g(u_j)} \cdot \ldots \cdot \frac{g(u_{i-b_2+1})}{g(u_{i-b_2})} = \frac{g(u_{i-b_2+1})}{g(u_j)}.$$

$\square$

3.4 Further Directions

There are two clear further directions following from these results:

(a) Is it possible to extend the three divisible weights case to three general weights?

(b) Is it possible to extend the three divisible weights case to more than three divisible weights?

Here we take some space to address the former. Unfortunately, it is not possible to extend our method of inequalities to address the case of 3 general weights, as illustrated by the following counterexample.

Let $k_1 = \frac{2}{3}, k_2 = \frac{1}{2}$, and $k_3 = \frac{1}{3}$. In this case we have the following inequalities:

$$\begin{aligned} 2q_1(1,0) \leq&\ 1\ < 2k_1 \\ 3q_2(0,2) \leq&\ 1\ < 3k_2 \\ 4q_3(0,0) \leq&\ 1\ < 4k_3 \\ q_2(0,0) + 2q_3(0,1) \leq&\ 1 < k_2 + 2k_3 \\ 2q_2(0,1) + q_3(0,2) \leq&\ 1 < 2k_2 + k_3 \end{aligned}$$

$$q_1(0,0) + 2q_3(1,0) \leq \ 1 < k_1 + 2k_3$$
$$q_1(0,1) + q_2(1,0) \leq \ 1 < k_1 + k_2$$
$$q_1(0,2) + q_2(1,0) \leq \ 1 < k_1 + k_2$$

If we were to try and use these to find weights β_I such that Equation (3.19) holds, we would find that $\lambda_{0,1} = \frac{2p_2}{p_3} \cdot \lambda_{0,0}$ and $\lambda_{0,2} = \frac{p_2}{2p_3} \cdot \lambda_{0,1} = \frac{p_2^2}{p_3^2} \cdot \lambda_{0,0}$, but also

$$\lambda_{1,0} = \frac{2p_1}{p_3} \cdot \lambda_{0,0}$$

and

$$\lambda_{1,0} = \frac{p_1}{p_2} \cdot \lambda_{0,1} + \frac{p_1}{p_2} \cdot \lambda_{0,2} = \left(\frac{2p_1}{p_3} + \frac{p_1 p_2}{p_3^2} \right) \lambda_{0,0}$$

which are inconsistent equations for most values of $\{p_i\}$.

However, this is not a counterexample to Conjecture 2, since it is still possible to find a defending distribution in this case. Although this particular set of inequalities does not have a solution (λ, β) that satisfies Equation (3.19), it is possible to find additional inequalities that allow us to find a solution. In fact, we prove a more general result, Theorem 9, in the next section. Unfortunately, the set of possible inequalities becomes more complicated, and the solution often depends on that values p_i, which prevented us from generalizing further along these lines.

3.5 Proof of Theorem 9

Here we prove of Theorem 9, restated below:

Theorem 9. *Suppose w is supported on three weights $k_1 > k_2 > k_3$ s.t. $k_1 \in (\frac{1}{2}, 1)$, $k_2 \in (\frac{1}{3}, \frac{1}{2}]$, and $k_1 + k_2 > 1$. Then for any distribution μ on A, it is possible to find a defending distribution.*

Proof. Once again let $A_i = \{a \in A \mid w(a) = k_i)$ and for any distribution μ on A, let $p_i = \mu(A_i)$ and μ_i be the conditional distribution of μ on A_i. As in the proof of Theorem 8, we consider pairs (r,t) such that $rk_1 + tk_2 \leq 1$ and let $c_{r,t} = \lfloor \frac{1-rk_1-tk_2}{k_3} \rfloor$ be the maximum number of elements from A_3 we can add to a set S already containing r elements from A_1 and t elements from A_2 without increasing the weight of the set to be greater than 1. In this case, we can define the set

$$\mathcal{R} = \{(0,0),(0,1),(0,2),(1,0)\}$$

and construct a defending distribution $\nu = \sum_{(r,t)\in\mathcal{R}} \lambda_{r,t}\nu_{r,t}$ as we did previously. Then $q_i(r,t)$ and $f_i(x_1,x_2,x_3)$ will again be defined as before.

Our proof will be split into 4 cases. Each case will use a subset of the following inequalities:

$I_1: \ (c_{0,0}+1)q_3(0,0) \leq 1 < (c_{0,0}+1)k_3$

$I_2: \ 3q_2(0,2) \leq 1 < 3k_2$

$I_3: \ 2q_1(1,0) \leq 1 < 2k_1$

$I_4: \ 2q_2(0,1) + (c_{0,2}+1)q_3(0,2) \leq 1 < 2k_2 + (c_{0,2}+1)k_3$

$I_5: \ q_2(0,0) + (c_{0,1}+1)q_3(0,1) \leq 1 < k_2 + (c_{0,1}+1)k_3$

$I_6: \ q_1(0,0) + (c_{1,0}+1)q_3(1,0) \leq 1 < k_1 + (c_{1,0}+1)k_3$

$I_7: \ q_1(0,1) + q_2(1,0) \leq 1 < k_1 + k_2$

$I_8: \ q_1(0,2) + q_2(1,0) \leq 1 < k_1 + k_2$

If it is the case that $c_{1,0} < c_{0,1}$, then we have

$I_9: \ q_1(0,1) + (c_{1,0}+1)q_3(1,0) \leq 1 < k_1 + (c_{1,0}+1)k_3$

If it is additionally the case that $c_{1,0} < c_{0,2}$, then we also have

$I_{10}: \ q_1(0,2) + (c_{1,0}+1)q_3(1,0) \leq 1 < k_1 + (c_{1,0}+1)k_3$

On the other hand, if $c_{1,0} = c_{0,1}$, then we have

$I_{11}:\ q_1(0,1) + c_{0,1} \cdot q_3(1,0) + 2q_2(0,1) + q_3(0,2) \le 2 < (k_1 + k_2) + (k_2 + (c_{0,1}+1)k_3)$

$I_{12}:\ q_2(0,0) + q_3(0,2) + c_{0,1} \cdot (2q_1(1,0) + 2q_2(0,1) + q_3(0,2)) \le 2 \cdot c_{0,1} + 1$

$< 2 \cdot c_{0,1}(k_1 + k_2) + (k_2 + (c_{0,1}+1)k_3)$

We first describe each case and its corresponding inequalities. For each case j with corresponding inequalities $\mathcal{I}_j$, we explicitly provide weights β_I^j and (non-normalized) values of $\lambda_{r,t}^j$ satisfying

$$\sum_{(r,t)\in\mathcal{R}} \sum_{i=1}^{3} \lambda_{r,t}^j p_i q_i(r,t) = \sum_{\mathcal{I}_j} \beta_I^j L_I. \tag{3.27}$$

We then provide a quick derivation of these inequalities under the conditions given by their corresponding cases, completing the proof.

The cases are as follows:

(a) $\frac{c_{0,2}-1}{2} \le c_{1,0} < c_{0,1}$

- In this case we use inequalities $I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8$, and I_9.
- The (non-normalized) weights are:

  $\lambda_{0,0} = \frac{2p_3^2}{(c_{0,2}+1)(c_{0,1}+1)}$ $\quad\beta_1 = \frac{2p_3^3}{(c_{0,2}+1)(c_{0,1}+1)(c_{0,0}+1)}$ $\quad\beta_6 = \frac{2p_1p_3^2}{(c_{0,2}+1)(c_{0,1}+1)}$

  $\lambda_{0,1} = \frac{2p_2p_3}{c_{0,2}+1}$ $\quad\beta_2 = \frac{p_2^3}{3}$ $\quad\beta_7 = (1-\delta)\cdot\frac{2p_1p_2p_3}{c_{0,2}+1}$

  $\lambda_{0,2} = p_2^2$ $\quad\beta_3 = \frac{(1-\delta)p_1^2p_3}{2(c_{0,2}+1)} + \frac{p_1^2p_2}{2}$ $\quad\beta_8 = p_1p_2^2$

  $\lambda_{1,0} = \frac{2(1-\delta)p_1p_3}{c_{0,2}+1} + p_1p_2$ $\quad\beta_4 = \frac{p_2^2p_3}{c_{0,2}+1}$ $\quad\beta_9 = \delta\cdot\frac{2p_1p_2p_3}{c_{0,2}+1}$

  $\beta_5 = \frac{2p_2p_3^2}{(c_{0,2}+1)(c_{0,1}+1)}$

  where δ is a parameter satisfying

  $$\frac{(c_{1,0}+1)(\lambda_{0,0}p_1 + \delta\lambda_{0,1}p_1)}{p_3} = \frac{(1-\delta)\lambda_{0,1}p_1 + \lambda_{0,2}p_1}{p_2}.$$

- We know such a δ exists, because $g(x) = \frac{(c_{1,0}+1)(\lambda_{0,0}+x\lambda_{0,1})}{p_3}$ and $h(x) =$

$\frac{(1-x)\lambda_{0,1}+\lambda_{0,2}}{p_2}$ are continuous functions on $[0,1]$ with

$$g(0) = \frac{(c_{1,0}+1)\lambda_{0,0}}{p_3} = \frac{2p_3(c_{1,0}+1)}{(c_{0,2}+1)(c_{0,1}+1)} < \frac{2p_3}{c_{0,2}+1} + p_2 = \frac{\lambda_{0,1}+\lambda_{0,2}}{p_2} = h(0)$$

and

$$g(1) = \frac{(c_{1,0}+1)(\lambda_{0,0}+\lambda_{0,1})}{p_3} = \frac{2p_3(c_{1,0}+1)}{(c_{0,2}+1)(c_{0,1}+1)} + \frac{2p_2(c_{1,0}+1)}{c_{0,2}+1} \geq p_2 = \frac{\lambda_{0,2}}{p_2} = h(1),$$

since in this case $\frac{c_{0,2}+1}{2} \leq c_{1,0}+1 < c_{0,1}+1$.

(b) $c_{1,0} < \frac{c_{0,2}-1}{2} < c_{0,2} \leq c_{0,1}$

- In this case we use inequalities $I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9$, and I_{10}.
- The (non-normalized) weights are:

$\lambda_{0,0} = \frac{2p_3^2}{(c_{0,2}+1)(c_{0,1}+1)}$ $\quad \beta_1 = \frac{2p_3^3}{(c_{0,2}+1)(c_{0,1}+1)(c_{0,0}+1)}$ $\quad \beta_6 = \frac{2p_1p_3^2}{(c_{0,2}+1)(c_{0,1}+1)}$

$\lambda_{0,1} = \frac{2p_2p_3}{c_{0,2}+1}$ $\quad \beta_2 = \frac{p_2^3}{3}$ $\quad \beta_7 = (1-\delta)\cdot\frac{2p_1p_2p_3}{c_{0,2}+1}$

$\lambda_{0,2} = p_2^2$ $\quad \beta_3 = \frac{1-\delta}{2}\left(\frac{2p_1^2p_3}{c_{0,2}+1} + p_1^2p_2\right)$ $\quad \beta_8 = (1-\delta)\cdot p_1p_2^2$

$\lambda_{1,0} = (1-\delta)\left(\frac{2p_1p_3}{c_{0,2}+1} + p_1p_2\right)$ $\quad \beta_4 = \frac{p_2^2p_3}{c_{0,2}+1}$ $\quad \beta_9 = \delta\cdot\frac{2p_1p_2p_3}{c_{0,2}+1}$

$\beta_5 = \frac{2p_2p_3^2}{(c_{0,2}+1)(c_{0,1}+1)}$ $\quad \beta_{10} = \delta\cdot p_1p_2^2$

where δ is a parameter satisfying

$$\frac{(c_{1,0}+1)(\lambda_{0,0}p_1 + \delta(\lambda_{0,1}p_1 + \lambda_{0,2}p_1))}{p_3} = \frac{(1-\delta)(\lambda_{0,1}p_1 + \lambda_{0,2}p_1)}{p_2}.$$

- We know such a δ exists, because $g(x) = \frac{(c_{1,0}+1)(\lambda_{0,0}+x(\lambda_{0,1}+\lambda_{0,2}))}{p_3}$ and $h(x) = \frac{(1-x)(\lambda_{0,1}+\lambda_{0,2})}{p_2}$ are continuous functions on $[0,1]$ with

$$g(0) = \frac{(c_{1,0}+1)\lambda_{0,0}}{p_3} = \frac{2p_3(c_{1,0}+1)}{(c_{0,2}+1)(c_{0,1}+1)} < \frac{2p_3}{c_{0,2}+1} + 2p_2 = \frac{\lambda_{0,1}+\lambda_{0,2}}{p_2} = h(0)$$

and $g(1) \geq 0 = h(1)$.

(c) $c_{1,0} = c_{0,1} = c$ and either $p_2 \geq p_1$ or $p_3 \geq p_1$

- In this case we use inequalities $I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8$, and I_{11}.
- The (non-normalized) weights are:

$\lambda_{0,0} = \frac{2p_3^2}{(c+1)}$ $\quad\beta_1 = \frac{2p_3^3}{(c+1)(c_{0,0}+1)}$ $\quad\beta_6 = \frac{2p_1p_3^2}{(c+1)}$

$\lambda_{0,1} = 2p_2p_3$ $\quad\beta_2 = \frac{p_2^3}{3}$ $\quad\beta_7 = 2(1-\delta)p_1p_2p_3$

$\lambda_{0,2} = p_2^2$ $\quad\beta_3 = (1-\delta)p_1^2p_3 + \frac{p_1^2p_2}{2}$ $\quad\beta_8 = p_1p_2^2$

$\lambda_{1,0} = 2(1-\delta)p_1p_3 + p_1p_2$ $\quad\beta_4 = p_2^2p_3 - 2\delta p_1p_2p_3$ $\quad\beta_{11} = 2\delta p_1p_2p_3$

$\beta_5 = \frac{2p_2p_3^2}{(c+1)}$

where $\delta = \frac{p_2}{2cp_2+2p_3}$ satisfies

$$\frac{(c+1)\lambda_{0,0}p_1 + c\delta\lambda_{0,1}p_1}{p_3} = \lambda_{1,0} = \frac{(1-\delta)\lambda_{0,1}p_1 + \lambda_{0,2}p_1}{p_2}.$$

- Note that since $\delta \leq \frac{1}{2c} \leq \frac{1}{2}$ and $\delta \leq \frac{p_2}{2p_3}$, we know $2\delta p_1p_2p_3 \leq p_1p_2p_3, p_1p_2^2$. Since either $p_2 \geq p_1$ or $p_3 \geq p_1$, this guarantees that $\beta_4 \geq 0$.

(d) $c_{1,0} = c_{0,1} = c$ and $p_2, p_3 < p_1$

- In this case we use inequalities $I_1, I_2, I_3, I_5, I_6, I_7, I_8$, and I_{12}.
- The (non-normalized) weights are:

$\lambda_{0,0} = \frac{2(1-\delta)p_3^2}{(c+1)} + \delta p_2p_3$ $\quad\beta_1 = \frac{2(1-\delta)p_3^3}{(c+1)(c_{0,0}+1)} + \delta\cdot\frac{p_2p_3^2}{c_{0,0}+1}$ $\quad\beta_6 = \frac{2(1-\delta)p_1p_3^2}{(c+1)} + \delta p_1p_2p_3$

$\lambda_{0,1} = 2(1-\delta)p_2p_3$ $\quad\beta_2 = \frac{p_2^3}{3}$ $\quad\beta_7 = 2(1-\delta)p_1p_2p_3$

$\lambda_{0,2} = p_2^2$ $\quad\beta_3 = (1-\delta)p_1^2p_3 + \frac{p_1^2p_2}{2} - c\delta p_2^2p_3$ $\quad\beta_8 = p_1p_2^2$

$\lambda_{1,0} = 2(1-\delta)p_1p_3 + p_1p_2$ $\quad\beta_5 = \frac{2(1-\delta)p_2p_3^2}{c+1}$ $\quad\beta_{12} = \delta p_2^2p_3$

where $\delta = \frac{1}{c+1}$ satisfies

$$2(1-\delta)p_1p_3 + p_1p_2 = 2(1-\delta)p_1p_3 + \delta p_1p_2(c+1).$$

- Note that since $p_1 > p_2, p_3$ and $1-\delta = 1-\frac{1}{c+1} = \frac{c}{c+1} = c\delta$, we have $\beta_3 > (1-\delta)p_2^2p_3 - c\delta p_2^2p_3 = 0$.

$\square$

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