An experimental walk in patterns, words, and partitions

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Ph.D. Thesis Defense

March 24, 2020

What is experimental mathematics?

- Broadly speaking, it is the philosophy that computers are a valuable tool that should be used extensively in mathematical research.
- Four forms of science experiments: Kantian/Baconian/Aristotelian/Galilian experiments–Peter Medawar (1915-1987) (*Experimental Mathematics in Action*)
- Computers used for: "experimenting", "solving", "deriving", "discovering", "conjecturing", "proving"...
- We use: Maple; Amarel cluter computing

Overview

An experimental walk in patterns, words, and partitions

I. Words with exactly one pattern 123
II. Increasing consecutive patterns in words
III. Relaxed partitions
IV. Systematic counting of restricted partitions and searching for new partition identities

1
Words with exactly one pattern 123

Definition

- A word $w = w_1 ... w_k$ is an ordered list of letters on some alphabet.
- A word $\pi = \pi_1\pi_2\cdots\pi_n$ contains a pattern $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ if at least one of the length-k sub-sequences of π reduces to σ. For example, 46793 contains the pattern 231.
- A word avoids the pattern σ if it does not contain it. For example, 13625 avoids the pattern 321.
- a word w is "in" $1^{l_1}\cdots n^{l_n}$ if w has l_i many i's in it $(1 \geq i \geq n)$. For example, 231113233 is a word in $1^3 2^2 3^4$.

A brief history

- Donald Knuth (1968), Rodica Simion, Richard Stanley, Herbert Wilf and others $\rightarrow$ forbidden patterns in permutations
- Alex Burstein (1998, under the guidance of Herbert Wilf) $\rightarrow$ forbidden patterns in words
- Doron Zeilberger (1998) & Vince Vatter (2006) $\rightarrow$ systematic, computer assisted enumeration of permutations avoiding a set of patterns
- Lara Pudwell (2010) $\rightarrow$ systematic $\cdots$ extension to words
- All of the above deal with pattern avoidence.

Noonan's theorem and our generalization

Noonan (1996)

The number of permutations with exactly one 321 pattern is equal to $\frac{3}{n}\binom{2n}{n+3}$.

- Alex Burstein (2011) provided a combinatorial proof of it.
- Dr. Zeilberger (2011) provided a shortened proof of the combinatorial proof.
- Dr. Z's proof idea: bijection between a permutation with exactly one pattern 321, denoted as $\pi_1 c \pi_2 b \pi_3 a \pi_4$ $(a < b < c)$, with the pair $(\pi_1 b \pi_2 a, c \pi_3 b \pi_4)$ where $\pi_1 b \pi_2 a$ is a 321-avoiding permutation of $\{1, ..., b\}$ and $c \pi_3 b \pi_4$ is a 321-avoiding permutation of $\{b, ..., n\}$.

We generalized this proof to apply it to words and arrived at:

Theorem 1

$A(l_1, ..., l_n)$: # of 123-avoiding words in $1^{l_1} \cdots n^{l_n}$.
$B(l_1, ..., l_n)$: # of words in $1^{l_1} \cdots n^{l_n}$ that contain the pattern 123 exactly once. We have:

$$B(l_1, ..., l_n) = \sum_{b=2}^{n-1} \sum_{j=0}^{l_b - 1} (A(l_1, ..., l_{b-1}, j+1) - A(l_1, ..., l_{b-1}, j))$$

$$\cdot (A(l_b - j, l_{b+1}, ..., l_n) - A(l_b - j - 1, l_{b+1}, ..., l_n)).$$

Proof idea:

- the set of words in $1^{l_1} \cdots n^{l_n}$ having exactly one pattern 123 $\leftrightarrow$ the set of "good" pairs (σ_1, σ_2), where σ_i is a 123-avoiding word.

Words in $1^r 2^r \cdots n^r$ with zero/one occurrences of 123

[Shar & Zeilberger, 2014]

- algorithm for finding a defining algebraic equation for the g.f. enumerating 123-avoiding words in $1^r 2^r \cdots n^r$
- for $r \leq 4$, a defining algebraic equation was found ($r > 5$ took too long to compute)

[2017]

- algorithm for finding a defining algebraic equation for the g.f. enumerating words in $1^r 2^r \cdots n^r$, with exactly one pattern 123.
- for $r \leq 3$, a defining algebraic equation was found ($r > 4$ took too long to compute)

(We all used the memory-intensive, and exponential time, Buchberger's algorithm for finding Gröbner bases.)

Idea of the extension

$h_r(x)$: the g.f. for words in $1^r2^r\cdots n^r$ (with weight $w \to x^{length(w)}$), with exactly one pattern 123.

$g_r(i,j)$: the g.f. for 123-avoiding words with any two letters having i and j occurrances, respectively, and the remaining letters each occurs r times.

Example: $h_2(x) = 2\cdot(g_2^{(0,0)}(x) - xg_2^{(0,1)}(x) - 1)(g_2^{(0,1)}(x) - xg_2^{(0,0)}(x))/x.$

Proof idea:
the coefficient of x^{2n} $(n \geq 0)$ on the right hand side is exactly the number of good pairs $(a\pi_3 b\pi_4, \pi_1 b\pi_2 c)$ $(2 \leq b \leq n-1)$, which equals to the number of words in $1^22^2\cdots n^2$ with exactly one pattern 123 (by the proof of **Theorem 1**).

The general formula

Theorem 2

$$h_r(x) = \frac{1}{x}\sum_{i=1}^{r}\left(g_r^{(0,\ i \bmod r)} - xg_r^{(0,\ i-1)} - \delta_{(i \bmod r,\ 0)}\right)$$

$$\cdot\left(g_r^{(0,\ (r+1-i) \bmod r)} - xg_r^{(0,\ r-i)} - \delta_{((r+1-i) \bmod r,\ 0)}\right).$$

Adding this to the system of $\binom{r+1}{2}$ equations for $g_r^{(i,j)}(x)$ (Shar & Zeilberger) :

$$g_r^{(i,j)}(x) = \delta_{i,0}\delta_{j,0} + x\sum_{t=0}^{r-1} g_r^{(i,t)}(x)g_r^{((r-t) \bmod r,\ (j-1) \bmod r)}(x)$$

$$+\sum_{m=0}^{i-1} x^{m+1}g_r^{(i-m,j-1)}(x)$$

Using Maple packages

Let $F = f_r(x) = h_r(x^{1/r})$. Using an existing procedure in Maple (Groebner[Basis]), we "solved" this system of equations for $r = 2$:

$$x^4\,(x+4)^2\,F^4 + 2\,x^3\,(x+4)\,(11\,x+23)\,F^3$$

$$-4\,x\left(3\,x^4 - 10\,x^3 - 97\,x^2 - 146\,x + 1\right)F^2$$

$$+(-168\,x^4 - 840\,x^3 - 744\,x^2 + 336\,x - 24)F$$
$$+144\,x^3\,(x+2) = 0.$$

- This takes a second to compute.
- We can also easily get a defining algebraic equation for $f_3(x)$, which takes about 20 seconds.
- The case when $r = 4$ already takes too long to compute (more than a month).

Asymptotics for $a_2(n)$ and $a_3(n)$

Let $a_r(n)$ be the number of words in $1^r 2^r \cdots n^r$ with exactly one pattern 123.

Having obtained the defining algebraic equations of the generating functions for $a_r(n)$ in the cases $r = 2$ and $r = 3$, Manuel Kauers kindly helped us in finding the asymptotics for our sequences $a_2(n)$ and $a_3(n)$:

$$a_2(n) = \frac{3(13 - \sqrt{21})}{49} \cdot \frac{1}{\sqrt{\pi}} \cdot 12^n \cdot n^{-3/2} \cdot (1 + O(n^{-1})),$$

$$a_3(n) = \frac{-7 + 6\sqrt{7}}{56} \cdot \frac{1}{\sqrt{\pi}} \cdot 32^n \cdot n^{-3/2} \cdot (1 + O(n^{-1})).$$

Using Maple packages

Using the *algtorec* procedure in the **SCHUTZENBERGER** package written by Doron Zeilberger, we get (for $r = 2$):

$$(36(1+n)(2+n)(1+2n)(3+2n)(18154800+23101940n+10635771n^2 +2093616n^3+147833n^4)+12(2+n)(3+2n)(1283329440+3700267618n +4200957553n^2+2408049238n^3+735936616n^4+113774584n^5 +6948151n^6)N+(282564806400+1066356868608n+1704365727480n^2 +1511140337906n^3+814587362081n^4+273775889012n^5+56080140110n^6 +6405068474n^7+312371129n^8)N^2-2(4+n)(11939685120+40890299130n +56943840213n^2+41794221496n^3+17488032270n^4+4183030930n^5 +531527997n^6+27792604n^7)N^3+8(1+n)(4+n)(5+n)(11+2n)(3742848 +7519914n+5241921n^2+1502284n^3+147833n^4)N^4)a_2(n)=0$$

Here we use $N^i a(n)$ to mean $a(n+i)$ $(i \geq 1)$.

2
Increasing consecutive patterns in words

joint with Doron Zeilberger

Definition and background

- Classical pattern avoidance: a word $\pi = \pi_1\pi_2\cdots\pi_n$ avoids a pattern $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ if none of the length-k sub-sequences of π reduces to σ. For example, 13625 avoids the pattern 321.
- Consecutive pattern avoidance: ...if none of the *consecutive* sub-sequences *("factors")* of π reduces to σ.
  For example, 13625 avoids the consecutive pattern 1324.
- Permutations avoiding consecutive patterns (Elizalde and Noy, 2003)
- Algorithmic approaches for permutations avoiding consecutive patterns (Nakamura, Baxter, and Zeilberger, 2011) $\rightarrow$ *Words*!

Outline

- How to count words that avoid the increasing consecutive pattern $12\cdots r$. (First due to Ira Gessel, new proof by tweaking the Goulden–Jackson cluster method)
- How to *efficiently* count words in $1^s2^s\cdots n^s$ that avoid the consecutive pattern $12\cdots r$.
- How to count words with a *specified number* of the consecutive pattern $12\cdots r$.

The Goulden–Jackson cluster method

Input: a *finite* alphabet $\{1, ..., n\}$, and a finite set of "bad words", B.
Output: the multivariable generating function:

$$F(x_1, \ldots, x_n) = \sum_{(m_1,\ldots,m_n)\in N^n} f(m_1, m_2, \ldots, m_n) x_1^{m_1} \cdots x_n^{m_n},$$

where $f(m_1, m_2, \ldots m_n)$ counts the words in $1^{m_1} \cdots n^{m_n}$ that never contain as consecutive subwords any memeber of B.

- Isn't this what we are looking for?
- Not exactly. This algorithm only allows a finite alphabet as input. Our alphabet will be arbitrary large and n will be symbolic.

Guessing using the Maple package DavidIan.txt (implementing the Goulden–Jackson cluster method)

Words avoiding pattern 123, for n=3 and 4:
GFpats({[1,2,3]},x,3,0) yields

$$\frac{1}{(1 - x_1 - x_2 - x_3 + x_1x_2x_3)}$$

GFpats({[1,2,3]},x,4,0) yields

$$\frac{1}{(1 - x_1 - x_2 - x_3 - x_4 + x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4 - x_1x_2x_3x_4)}$$

Continuing in this fashion, it's straightforward to guess, for general n, we have:

$$F_3(x_1, x_2, \ldots, x_n) = \frac{1}{1 - e_1 + e_3 - e_4 + e_6 - e_7 + e_9 - e_{10} + \ldots}$$

Words in $\{1, 2, \ldots, n\}$ avoiding $12\cdots r$

Doing the analogous guessing for the consecutive patterns 1234 and 12345, we arrived at the following general statement, for general consecutive pattern $12\cdots r$:

Theorem 1 (Gessel, 1977)

For $n \geq 1$, $r \geq 2$, the generating function for words in the alphabet $\{1, 2, \ldots, n\}$ avoiding the consecutive pattern $12\cdots r$ is:

$$F_r(x_1, x_2 \ldots, x_n) = \frac{1}{1 - e_1 + e_r - e_{r+1} + e_{2r} - e_{2r+1} + e_{3r} - e_{3r+1} + \cdots}$$

This appeared in Ira Gessel's Ph.D. thesis, thanks to Justin Troyka for pointing it out after we posted the article on arXiv. We proved it by tweaking the Goudlen–Jackson cluster method.

The Goulden–Jackson cluster method

$$weight(w) := \prod_{i=1}^{n} x_{w_i}$$

Marked word: a marked word is simply a word with some factors marked. For example, (53875, {[1,3], [2,4]}) is a marked word with factors 538, 387 marked.

$$\overline{weight}(w, F) := (-1)^{|F|} \prod_{i=1}^{n} x_{w_i}$$

The weight of all good words = the weight of all marked words!!
Therefore, $\overline{weight}(53875, \{[1,3],[2,4]\}) = (-1)^2 x_5^2 x_3 x_8 x_7 = x_5^2 x_3 x_8 x_7$

$$\overline{weight}(M) = \frac{1}{1 - (x_1 + \cdots + x_n) - \overline{weight}(C)}$$

C: set of all clusters. Cluster example: (39759, $\{[1,3],[2,4],[3,5]\}$)

Tweaking the Goulden–Jackson cluster method

- If we use the original Goulden–Jackson cluster method to figure out $\overline{weight}(C)$, we would need to solve a system of $\binom{n}{r}$ equations.
- In our case, $\overline{weight}(C)$ is a summation of multivariate monomials on $\{x_1, x_2 \cdots x_n\}$ where the exponent of x_i is 0 or 1 ($1 \leq i \leq n$). Our goal is to figure out the coefficient of these monomials. By separating the clusters for $123 \cdots k$ into $(r-1)$ categories based on where the second mark is, we get the following:

Lemma:

For $k > r$, $coeff(x_1 x_2 \cdots x_k)$
$= -coeff(x_2 x_3 \cdots x_k) - coeff(x_3 x_4 \cdots x_k) - \cdots - coeff(x_r x_{r+1} \cdots x_k)$
$= -coeff(x_1 x_2 \cdots x_{k-1}) - \cdots - coeff(x_1 x_2 \cdots x_{k-r+1})$

From this it's easy to deduce:
$\overline{weight}(C) = -e_r + e_{r+1} - e_{2r} + e_{2r+1} - e_{3r} + e_{3r+1} - \cdots$
Theorem 1 directly follows.

Efficient computations

Recall **Theorem 1**: For $n \geq 1$, $r \geq 2$, the generating function for words in the alphabet $\{1, 2, \ldots, n\}$ avoiding the consecutive pattern $12 \cdots r$ is:

$$F_r(x_1, x_2 \ldots, x_n) = \frac{1}{1 - e_1 + e_r - e_{r+1} + e_{2r} - e_{2r+1} + e_{3r} - e_{3r+1} + \cdots}$$

$$= \sum_{(m_1, \ldots, m_n) \in N^n} f_r(m_1, \ldots, m_n) x_1^{m_1} \cdots x_n^{m_n}$$

where $f_r(m_1, \ldots, m_n)$ is the number of words in $1^{m_1} 2^{m_2} \cdots n^{m_n}$ that avoid the consecutive pattern $12 \cdots r$.

Observe that $f_r(m_1, m_2 \ldots, m_n)$ is symmetric because $F_r(x_1, x_2 \ldots, x_n)$ is symmetric.

Efficient computations

Fundamental Recurrence: Let $f_r(\mathbf{m})$ be the number of words in $1^{m_1}2^{m_2}\cdots n^{m_n}$ (where $\mathbf{m} = (m_1, \ldots, m_n)$) that avoid the consecutive pattern $12\cdots r$. Also let V_i be the set of 0-1 vectors of length n with i 1's, then

$$
\begin{aligned}
f_r(\mathbf{m}) = \sum_{\mathbf{v}\in V_1} f_r(\mathbf{m}-\mathbf{v}) \; - \sum_{\mathbf{v}\in V_r} f_r(\mathbf{m}-\mathbf{v}) \\
+ \sum_{\mathbf{v}\in V_{r+1}} f_r(\mathbf{m}-\mathbf{v}) \; - \sum_{\mathbf{v}\in V_{2r}} f_r(\mathbf{m}-\mathbf{v}) \\
+ \sum_{\mathbf{v}\in V_{2r+1}} f_r(\mathbf{m}-\mathbf{v}) \; - \sum_{\mathbf{v}\in V_{3r}} f_r(\mathbf{m}-\mathbf{v}) + \cdots
\end{aligned}
$$

Efficient computations-permutations

$a_r(n) := f_r(1^n)$, the number of *permutations* on $\{1, 2, \ldots, n\}$ that avoid the consecutive pattern $1 \cdots r$.

Fundamental Recurrence $\Rightarrow$

$$\begin{aligned} a_r(n) = na_r(n-1) - \binom{n}{r} a_r(n-r) + \binom{n}{r+1} a_r(n-r-1) \\ -\binom{n}{2r} a_r(n-2r) + \binom{n}{2r+1} a_r(n-2r-1) \\ -\binom{n}{3r} a_r(n-3r) + \binom{n}{3r+1} a_r(n-3r-1) - \cdots \end{aligned}$$

This recurrence goes back to David and Barton (1962). Note that it takes $O(n^2)$ steps to compute $a_r(n)$ using the recurrence above.

Efficient computations-two occurrences of each letter

$b_r(n) := f_r(2^n)$, the number of words with 2 occurrences of each of $1, 2, \ldots, n$ avoiding the pattern $1 \cdots r$.
Plugging in $f_r(2^n)$ into the **Fundamental Recurrence**, we are forced to consider $f_r(2^\alpha 1^\beta)$.

$$B_r(\alpha, \beta) := f_r(2^\alpha 1^\beta)$$

Using symmetry, we get the following recurrence for $B_r(\alpha, \beta)$.

$$B_r(\alpha, \beta) = \alpha B_r(\alpha - 1, \beta + 1) + \beta B_r(\alpha, \beta - 1)$$
$$- \sum_{i_1+i_2=r} \binom{\alpha}{i_1}\binom{\beta}{i_2} B_r(\alpha - i_1, \beta - i_2 + i_1) + \sum_{i_1+i_2=r+1} \binom{\alpha}{i_1}\binom{\beta}{i_2} B_r(\alpha - i_1, \beta - i_2 + i_1)$$
$$- \sum_{i_1+i_2=2r} \binom{\alpha}{i_1}\binom{\beta}{i_2} B_r(\alpha - i_1, \beta - i_2 + i_1) + \sum_{i_1+i_2=2r+1} \binom{\alpha}{i_1}\binom{\beta}{i_2} B_r(\alpha - i_1, \beta - i_2 + i_1)$$
$$- \cdots$$

$b_r(n) = B_r(n, 0)$
We can compute $b_r(n)$ in cubic time.

Efficient computations-three/four occurrences of each letter

In similar fashion, we can compute $f_r(3^n)$ in quartic time and $f_r(4^n)$ in quintic time. And we easily obtained 40 and 20 terms of them respectively, for each $3 \leq r \leq 9$.

For example, the first 15 terms of $f_7(3^n)$ are:
1, 20, 1680, 369600, 168168000, 137225088000, 182499151015439, 369333660414653745, 1080107104118231632500, 4384231121059173932562000, 23913914175434871142808715000, 170693577054027116430454774306800, 1559452501977701854639515593122328400, 17896181334529134150000549426290350987991, 253852106096411581819653232416307542549727141.

Keeping track of the number of occurrences

Let $weight(w) := (t-1)^{|F|} \cdot \prod_{i=1}^{k} x[w_i]$, where F is the set of marks in w. We have the following:

Theorem 3

The generating function for words in $1^{m_1}2^{m_2}\cdots n^{m_n}$ and exactly k consecutive patterns $12\cdots r$ is:

$$G_r(x_1,\ldots,x_n;t) = \frac{1}{1 - e_1 - \sum_{k=r}^{n} P_k^{(r)}(t) e_k} \quad .$$

Definition

For any integer $k \geq 1$ and $r \geq 2$, $P_k^{(r)}(t)$ is defined as follows.
If $k < r$, then it is 0. If $k = r$ then it is $t-1$, and if $k > r$ then

$$P_k^{(r)}(t) \;=\; (t-1)\sum_{i=1}^{r-1} P_{k-i}^{(r)}(t) \quad .$$

Future work

Sergi Elizalde, Marc Noy
Clusters, generating functions and asymptotics for consecutive patterns in permutations
Adv. Appl. Math. (2012)
https://doi.org/10.1016/j.aam.2012.08.003

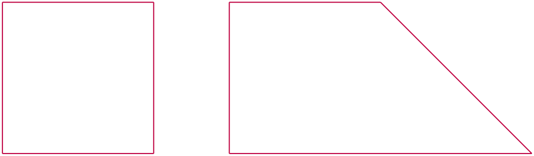
Tim Dwyer, Sergi Elizalde
Wilf equivalence relations for consecutive patterns
Adv. Appl. Math. (2018)
https://doi.org/10.1016/j.aam.2018.04.007

Sergi's suggestion: One may be able to derive a definition of consecutive-Wilf equivalence of patterns in words (as opposed to in permutations) and study equivalence classes there.

3
Relaxed partitions

Definition

- A partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, ..., \lambda_k$ whose sum is equal to n. For example, $(4, 4, 2, 1)$ is a partition of 11.
- A relaxed partition of a positive integer n is a finite sequence of positive integers $\lambda_1, \lambda_2, ..., \lambda_k$ $(\lambda_i - \lambda_{i+1} \geq r)$ whose sum is equal to n. We also call this partition an r-partition of n. For example, $(3, 4, 2, 1)$ a (-1)-partition of 10. Partitons into distinct parts are 1-partitions.



Generating function for relaxed partitions

Q: For a fixed r, how many r-partitions of integer n do we have?

We used Maple to program $NPr(n, r)$: # of r-partitions of n for any n and r. The first 20 terms of $NPr(n, -1)$ (# of (-1)-partitions of n) are:

1,2,4,7,13,23,41,72,127,222,388,677,1179,2052,3569,6203,10778,
18722,32513,56455

Now, can we find a generating function for a given r? The answer turned out to be yes! We typed the above sequence produced by $NPr(n, -1)$ into the **OEIS**, and found that its generating function seemed to be the reciprocal of

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{k^2}}{(q;q)_k} \quad .$$

Generating function for relaxed partitions

Theorem 1 [Zeilberger, 2018]

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{k^2}}{(q;q)_k} \sum_{n=0}^{\infty} NPr(n,-1)q^n = 1$$

- Doron Zeilberger provided an elegant bijective proof of it, and the proof can be easily generalized to derive the following:

Theorem 2 (Generalization of Theorem 1)

Given a negative integer r, the generating function for the number of r-partitions of n is the reciprocal of

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{k(2+(1-r)(k-1))/2}}{(q;q)_k}$$

Relaxed partitions with restrictions

What if we restrict the first part to be M and the number of parts to be exactly N? Let us call this generating function $F(M, N, r, q)$.

It is not hard to come up with a recurrence relation for $F(M, N, r, q)$:

$$F(M, N, r, q) = q^M \sum_{M_1=1}^{M-r} F(M_1, N-1, r, q)$$

Conjecture and proof using Maple

Knowing the reccurence relation, we programmed in Maple and found the following:

Typing $[\mathbf{seq}(\mathbf{F}(\mathbf{M1}, \mathbf{1}, -\mathbf{1}, \mathbf{1}), \mathbf{M1} = \mathbf{1}..\mathbf{20})]$; yields

$$[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

To guess a polynomial for this sequence, type:
$\mathbf{GuessPol}([\mathbf{seq}(\mathbf{F}(\mathbf{M1}, \mathbf{1}, -\mathbf{1}, \mathbf{1}), \mathbf{M1} = \mathbf{1}..\mathbf{20})], \mathbf{M}, \mathbf{1})$;
Not surprisingly, it yields the constant polynomial 1.

Now try $N = 2$. Typing $[\mathbf{seq}(\mathbf{F}(\mathbf{M1}, \mathbf{2}, -\mathbf{1}, \mathbf{1}), \mathbf{M1} = \mathbf{1}..\mathbf{20})]$; yields

$$[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]$$

$\mathbf{GuessPol}([\mathbf{seq}(\mathbf{F}(\mathbf{M1}, \mathbf{2}, -\mathbf{1}, \mathbf{1}), \mathbf{M1} = \mathbf{1}..\mathbf{20})], \mathbf{M}, \mathbf{1})$; yields

$$M + 1$$

Conjecture and proof using Maple

Also not a surpise. Moving right along,
$\mathbf{GuessPol([seq(F(M1, 3, -1, 1), M1 = 1..20)], M, 1)};$ yields

$$\frac{(M+4)(M+1)}{2}$$

$\mathbf{GuessPol([seq(F(M1, 4, -1, 1), M1 = 1..20)], M, 1)};$ yields

$$\frac{(M+6)(M+5)(M+1)}{6}$$

$\mathbf{GuessPol([seq(F(M1, 5, -1, 1), M1 = 1..20)], M, 1)};$ yields

$$\frac{(M+8)(M+7)(M+6)(M+1)}{24}$$

$\mathbf{GuessPol([seq(F(M1, 6, -1, 1), M1 = 1..20)], M, 1)};$ yields

$$\frac{(10+M)(9+M)(8+M)(7+M)(M+1)}{120}$$

Conjecture and proof using Maple

Without much effort, one can conjecture the following:

$$F(M,N,-1,1) = \frac{(M+1)(M+2N-2)!}{(N-1)!(M+N)!} \quad .$$

With similar experimentation with $r=-2$, one can conjecture that

$$F(M,N,-2,1) = \frac{(M+2)(M+3N-2)!}{(N-1)!(M+2N)!} \quad .$$

Comparing these two guesses, one can easily conjecture the formula for a general r:

$$F(M,N,r,1) = \frac{(M-r)(M+(1-r)N-2)!}{(N-1)!(M-rN)!} \quad .$$

Conjecture and proof using Maple

Now, how do we prove this conjecture?
Recall that we programmed $F(M, N, r, q)$ using the recurrence relation

$$F(M, N, r, q) = q^M \sum_{M_1=1}^{M-r} F(M_1, N-1, r, q)$$

and the initial condition $F(M, 1, r, q) = q^M$.

- Note that $F(M, N, r, q)$ can be fully defined by this information.
- In other words, if we have found a formula that satisfies this recurrence relation and initial condition, then it *is* the formula for $F(M, N, r, q)$. This also applies to our current case when $q = 1$.
- We used Maple to verify both this equation and the intial condition are satisfied. So our conjectured formula $F(M, N, r, 1)$ was proved.

Conjecture and proof using Maple

Theorem 3

$$F(M,N,r,1) = \frac{(M-r)(M+(1-r)N-2)!}{(N-1)!(M-rN)!}$$
$$= \binom{M+(1-r)N-2}{N-1} + r\binom{M+(1-r)N-2}{N-2}$$

Now the next step is to try to conjecture a formula for $F(M,N,r,q)$.

Can we find a pattern for $F(M, N, r, q)$?

This turns out to be not so easy. Below are the guesses for $r = -1$ and $N <= 5$:

$\mathbf{qGuessPol([seq(F(M1, 1, -1, q), M1 = 1..20)], M, q, 1)}$; yields

$$q^M$$

$\mathbf{qGuessPol([seq(F(M1, 2, -1, q), M1 = 1..20)], M, q, 1)}$; yields

$$\frac{q^{M+1}(q^{M+1} - 1)}{(q - 1)}$$

$\mathbf{qGuessPol([seq(F(M1, 3, -1, q), M1 = 1..20)], M, q, 1)}$; yields

$$\frac{q^{M+2}(q^{M+3} + q^2 - q - 1)(q^{M+1} - 1)}{(q - 1)^2(q + 1)}$$

Can we find a pattern for $F(M, N, r, q)$?

$\mathbf{qGuessPol([seq(F(M1, 4, -1, q), M1 = 1..20)], M, q, 1)}$; yields

$$\frac{q^{M+3}(q^{2M+8} + q^{M+7} - q^{M+5} - q^{M+4} - q^{M+3} + q^6 - 2q^4 - q^3 + 2q + 1)(q^{M+1} - 1)}{(q-1)^3(q+1)(q^2+q+1)}$$

$\mathbf{qGuessPol([seq(F(M1, 5, -1, q), M1 = 1..20)], M, q, 1)}$; yields

$$\frac{q^{M+4}(q^{M+5} + q^4 - q - 1)(q^{2M+10} - q^{M+4} - q^{M+3} + q^8 - q^5 - 2q^4 + 2q + 1)(q^{M+1} - 1)}{(q-1)^4(q+1)^2(q^2+1)(q^2+q+1)}$$

Again, we can **prove** that they are true by using the recurrence relation. Note that, although the formulas above look like rational functions, they are in fact polynomials. It would be very nice to find a general pattern for $F(M, N, r, q)$.

Some observations

$(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$ (q-Pochhammer Symbol)

Drew Sills' Observation: $F(M,N,-1,q)$ has denominator $(q;q)_N$ and a numerator of degree $N(M+N-1)$. Thus it is plausible that the numerator is a (possibly alternating) sum of polynomials that are a power of q times a linear transformation of the Gaussian polynomial: $G(M,N) := GP(2N+M-1,N)$.

$$GP(m,r) := \frac{(q^{m-r+1};q)_r}{(q;q)_r}$$

So far we have not made much progress in this, but this led to an interesting discovery.

Some observations

$$\begin{cases} G(2,1) = \underline{q^2 + q + 1} \\ F(2,2,-1,q) = \underline{q^5 + q^4 + q^3} \end{cases}$$

$$\begin{cases} G(2,2) = \underline{q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q} + 1 \\ F(2,3,-1,q) = \underline{q^9 + q^8 + 2q^7 + 2q^6 + 2q^5 + q^4} \end{cases}$$

$$\begin{cases} G(2,3) = \underline{q^{12} + q^{11} + 2q^{10} + 3q^9 + 4q^8 + 4q^7 + 5q^6 + 4q^5} + 4q^4 \\ \qquad +3q^3 + 2q^2 + q + 1 \\ F(2,4,-1,q) = \underline{q^{14} + q^{13} + 2q^{12} + 3q^{11} + 4q^{10} + 4q^9 + 5q^8} \\ \qquad \underline{+4q^7} + 3q^6 + q^5 \end{cases}$$

In general, the conjecture is that we can predict the first $(2N + 2M - 4)$ terms of $F(M, N, -1, q)$ using the first $(2N + 2M - 4)$ terms of $G(M, N - 1)(N \geq 3)$.

Connection to other combinatorial objects

- There is a direct connection between $F(M, N, -1, 1)$ and Catalan's triangle: $F(M, N, -1, 1) = C(M + N - 1, N - 1)$.
- There also seems to be a bit of connection between the standard Young Tableau and $F(M, N, -1, 1)$. For example, typing the sequence $[seq(F(5, N, -1, 1), N = 1..20]$ into **OEIS**, we will found it can also represent the number of standard Young Tableau of shape $(N + 3, N - 2)$. (A003517)
- We conjecture that $F(M, N, -1, 1)$ is equal to the number of standard Young Tableau of shape $(N + \lceil M/2 \rceil, N - \lfloor M/2 \rfloor)$.

4
Systematic counting of restricted partitions and searching for new partition identities

joint with Matthew C. Russell and Doron Zeilberger

Some fascinating partition identities

q-Pochhammer Symbol: $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$
$(a;q)_\infty := \prod_{j\geq 0}(1-aq^j)$

Euler's Odd Distinct identity (1748):

$$\prod_{all\ i}(1+q^i) = \prod_{i\ odd}\frac{1}{1-q^i} = \frac{1}{(q;q^2)_\infty}$$

distinct parts → odd parts

Rogers–Ramanujan identities (1894):

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}$$

adjacent parts differ by at least 2 → parts 1 or 4 mod 5

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$$

adjacent parts differ by at least 2, smallest part at least 2 → parts 2 or 3 mod 5

Motivation and background

$p(n)$: number of partitions of n.
$P(n,m)$: number of partitions of n with the largest part m.

$$p(n) = \sum_{m=1}^{n} P(n,m).$$

$$P(n,m) = \sum_{m'=1}^{m} P(n-m,m') \quad , \quad n \geq m \geq 1$$

$$\Rightarrow P(n,m) = P(n-1,m-1) + P(n-m,m).$$

The last equation gives us an efficient way (quadratic in time and memory) to compute a table for $p(n)$. But what if we want to count not just any partition efficiently, but partitions with some restrictions? What if we only want to count partitions whose adjacent parts differ by at least 2 (Rogers–Ramanujan)?

Definition

- A **pattern** is a list $a = [a_1, a_2, \ldots, a_r]$ $(r \geq 1)$ of non-negative integers.
- A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ **contains** the pattern $a = [a_1, a_2, \ldots, a_r]$ if there exists $1 \leq i \leq k - r$ such that:
  $\lambda_i - \lambda_{i+1} = a_1, \quad \lambda_{i+1} - \lambda_{i+2} = a_2, \ldots, \lambda_{i+r-1} - \lambda_{i+r} = a_r.$
- For example, $(7, 6, 5, 4, 4)$ contains the pattern $[1]$, the pattern $[0]$, the pattern $[1, 1]$, the pattern $[1, 0]$, and so on.
- A partition λ **avoids** (globally) the pattern if it does **not** contain the pattern.
- A partition λ **avoids** (globally) the set of patterns A, if it avoids *every* pattern in A. For example, partitions whose adjacent parts differ by at least 2 is equivalent to partitions that avoid $\{[0], [1]\}$.
- A partition λ contains a pattern *at the beginning* if we begin from the largest part of the partition and the pattern immediately appears. For example, (7,6,5,4,4) contains the pattern $[1, 1]$ at the beginning.

Idea of our initial algorithm

- $P_{[1,1,1]}(n,m)$: number of partitions of n with largest part m, avoiding $[1,1,1]$.
- $P'_{[1,1,1]}(n,m)$: the number of partitions of n with largest part m, avoiding the pattern $[1,1,1]$, and in addition, avoiding the pattern $[1,1]$ at the very beginning.
- $P''_{[1,1,1]}(n,m)$: number of partitions of n with largest part m, avoiding $[1,1,1]$, and *in addition*, avoiding the pattern $[1]$ at the beginning.

$$P_{[1,1,1]}(n,m) = \sum_{\substack{1\le m'\le m \\ m'\neq m-1}} P_{[1,1,1]}(n-m,m') + P'_{[1,1,1]}(n-m,m-1)$$

$$P'_{[1,1,1]}(n,m) = \sum_{\substack{1\le m'\le m \\ m'\neq m-1}} P_{[1,1,1]}(n-m,m') + P''_{[1,1,1]}(n-m,m-1)$$

$$P''_{[1,1,1]}(n,m) = \sum_{\substack{1\le m'\le m \\ m'\neq m-1}} P_{[1,1,1]}(n-m,m')$$

This can be made into a quadratic time and memory algorithm to compute a table for $P_{[1,1,1]}(n)$. The same is true for $P_A(n)$ for a general (finite) A.

Going beyond...

What if we want to count partitions with more specific restrictions, for example, not just globally, but also based on congruence conditions?

- Recall one side of Schur's celebrated 1926 theorem: partitions of m into parts with minimal difference 3 and with no consecutive multiples of 3.
- And how about the a more complicated Kanade–Russell conjecture:
  (1) No parts repeat.
  (2) Adjacent parts do not differ by 1 if the smaller part is even.
  (3) A sub-partition of type $(2j+4)+(2j+2)+2j$ is not allowed.
  (4) A sub-partition of type $(2j+4)+(2j+2)+(2j+1)$ is not allowed.
  (5) A sub-partition of type $(2j+4)+(2j+3)+(2j+1)$ is not allowed.
  (6) Smallest part is at least 3.

Definition refined

- m, n, A: same as before (A is the set of patterns to avoid globally.)
- *Mod*: (loosely speaking) the list of patterns to avoid according to mod conditions of the largest part of a sub-partition

  Examples:
  1. $Mod = [\{[1,1]\},\{[2]\},\{\}] \to$ forbidding sub-partition of type $(3j+3)+(3j+2)+(3j+1)$ and type $(3j+4)+(3j+2)$
  2. $Mod = [\{[0]\},\{[0,0]\}] \to$ even parts are not allowed to repeat, and odd parts can appear at most twice
- B: the set of patterns to avoid at the beginning of the partition
- I: the set of sub-partitions to avoid (we call this "initial conditions")

In light of these new notations, we have:

Schur

(1) Parts with minimal difference 3 $\rightarrow A = \{[0],[1],[2]\}$
(2) No sub-partition of type $(3j+3)+3j \rightarrow Mod = [\{[3]\},\{\},\{\}]$

Kanade–Russell

(1) No parts repeat $\rightarrow A = \{[0]\}$
(2) A sub-partition of type $(2j+1)+2j$ is not allowed.
(3) A sub-partition of type $(2j+4)+(2j+2)+2j$ is not allowed.
(4) A sub-partition of type $(2j+4)+(2j+2)+(2j+1)$ is not allowed.
(5) A sub-partition of type $(2j+4)+(2j+3)+(2j+1)$ is not allowed.

(2),(3),(4),(5) $\rightarrow Mod = [\{[2,2],[2,1],[1,2]\},\{[1]\}]$

(6) Smallest part is at least 3 $\rightarrow I = \{[1],[2]\}$

Generalized algorithm

Let $GP(m, n, A, Mod, B, I)$ be the number of partitions of n, with largest part m, and the restrictions A, Mod, B, I.

(1) If $m > n$, return 0. If $m = n$, return 1.

(2) Check if m is equal to the largest part of any of the forbidden sub-partitions in I: if so, and if the forbidden sub-partition is just [m] then return 0, otherwise we add the underlying partition pattern to B and we have a set of new beginning restrictions B'.

Generalized algorithm

(3) If $Mod = \{\}$, then by chopping off the largest part we get the recurrence:

$$GP(n, m, A, Mod, B, I) = \sum_{\substack{1 \le m' \le m \\ [m-m'] \notin A \cup B'}} GP(n-m, m', A, Mod, B'', I).$$

Note the "valid" m' will be those such that the singleton $[m-m']$ is not in the forbidden patterns (either globally or at the beginning). B'' is the set of new beginning restrictions, obtained from $A \cup B'$ by chopping off the difference $m - m'$ from the patterns in $A \cup B'$.

m $\quad m'$ $\qquad\qquad [a, b] \in A \cup B'$

$\quad a \quad b$

$\downarrow$

B''

Generalized algorithm

(4) If $Mod \neq \{\}$, let the length of Mod be k. If $m \equiv i \pmod{k}$ then we get the recurrence:

$$GP(n, m, A, Mod, B, I) = \sum_{\substack{1 \leq m' \leq m \\ [m-m'] \notin A \cup B' \cup Mod[i+1]}} GP(n-m, m', A, Mod, B'', I).$$

"Valid" m' will be those such that the singleton $[m - m']$ is not in the forbidden patterns (either globally or at the beginning or according to the mod condition).

B'' is the set of new beginning restrictions, obtained from $A \cup B' \cup Mod[i+1]$ by chopping off the difference $m - m'$ from the patterns in $A \cup B' \cup Mod[i+1]$.

Close analysis is still to be done but I think this algorithm can also be made quadratic in time and memory to compute a table for $p_{A,Mod,B,I}(n)$.

A little (very incomplete) history of searching for identities

- 1894: Rogers–Ramanujan identities first published (MacMahon verified by hand, calculating 89 terms)
- 1952: “Slater list”
- 1970: Andrews computer search
- 1988: Capparelli identities (conjectured from VOA, proved by Andrews and many others later)
- 2009: Mc Laughlin, Sills and Zimmer computer search
- 2014+: Kanade–Russell computer search
- 2014: Nandi’s conjectures (obtained from Lie algebra, have been proved last year!)

Preliminaries

- "Sum side": a generating function that counts the pattern-avoiding partitions that we are currently interested in (according to *A*, *Mod* and *I*). It may or may not have an analytic (multi-)sum.
- "Product side": the side with infinite products. We use Frank Garvan's **qseries** Maple package to "factor" the generating function from the "sum side" into infinite products.
- We use a "list notation" to denote a "product side", for example $[-2,-1,0,1,0]$ denotes $\frac{(q^4;q^5)_\infty}{(q;q^5)_\infty^2(q^2;q^5)_\infty}$. So if the list has only -1 and 0 in it, that means the "product side" satisfies certain congruence conditions.

  For example, $[-1,0,0,-1,0]$ denotes $\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}$, that is, the parts are 1 or 4 modulo 5, which is a famous Rogers–Ramanujan "product side".

Discoveries

Needless to say, we discovered many old identities, like Gordon, Andrews-Bressoud, Capparelli, among many others. But many of them are new. I will present a very incomplete list here.

A, Mod, I $\rightarrow$ Product Side

(1) $\{\}$, $[\{[1],[2]\},\{[0],[2],[3]\},\{\}]$, $\{\}$ $\rightarrow$ 0, 1, 3, 6, 7, 8, 9, 11 *mod* 12

(2) $\{\}$, $[\{\},\{[1],[2]\},\{[0],[2],[3]\}]$, $\{[1],[2]\}$ $\rightarrow$ 0, 3, 4, 5, 6, 9, 11 *mod* 12

(3) $\{[1]\}$, $[\{[0],[3]\},\{\},\{\}]$, $\{\}$ $\rightarrow$ 1, 2, 4, 6, 8, 10, 11 *mod* 12

(4) $\{[1]\}$, $[\{\},\{[0],[3]\},\{\}]$, $\{[1]\}$ $\rightarrow$ 0, 2, 3, 4, 6, 9, 10 *mod* 12

(5) $\{[1]\}$, $[\{\},\{\},\{[0],[3]\}]$, $\{[1]\}$ $\rightarrow$ 0, 2, 3, 6, 8, 9, 10 *mod* 12

(6) $\{[1]\}$, $[\{\},\{[0],[3]\},\{[0],[3]\}]$, $\{[1]\}$ $\rightarrow$ 0, 2, 3, 6, 9, 10 *mod* 12

A, Mod, I → Product Side

(7) {}, [{[0, 1]}, {[2], [1, 1]}], {} → 0, 1, 2, 3, 6, 7, 8, 9, 10 *mod* 12

(8) {}, [{[0, 1], [1, 2]}, {[0], [1, 1], [2, 2]}], {} → 0, 1, 3, 4, 7, 8, 9, 10 *mod* 12

(9) {[1, 0], [1, 1, 1]}, [ ], {[1]} → 0, 2, 3, 4, 5, 6, 8, 9, 11 *mod* 12

(10) {[1, 1], [0, 0, 0], [1, 0, 1], [1, 0, 0, 1]}, [ ], {} → 1, 2, 3, 5, 7, 9, 10, 11 *mod* 12

From (9), we obtained its "companion identity" by hand:
(9) {[0, 1], [1, 1, 1]}, [ ], {[1,1], [3,2,1]} → 0, 1, 3, 4, 6, 7, 8, 9, 10 *mod* 12

Revisit (10): $\{[1, 1], [0, 0, 0], [1, 0, 1], [1, 0, 0, 1]\}$, $[\ ]$, $\{\} \to 1, 2, 3, 5, 7, 9, 10, 11$ *mod* 12

- Observe that the "sum side" of (10) is equivalent to:
  –At most 3 occurrences of every part
  –For all i, not allowed to have i, $i+1$, $i+2$ in the partition
- This seems to generalize to an infinite family:
  – At most k occurrences of any given part
  – For all i, not allowed to have $i, i+1, \ldots, i+k-1$ all as parts in the partition (a.k.a. avoiding k-sequences).

Conjecture

Fix $k \geq 3$. Let $A_k(n)$ be the number of partitions of n avoiding k-sequences, and with at most k occurrences of of any given part. Then,

$$\sum_{n\geq 0} A_k(n)q^n = \frac{(q^{k+1}; q^{k+1})_\infty (q^{k(k+1)/2}; q^{k(k+1)})_\infty}{(q;q)_\infty}.$$

Note that when $k = 2$, this is a special case of the Andrews–Bressoud identities.

Additional identities

The "product sides" of (1) – (10) all correspond to partitions whose parts satisfy certain congruence conditions, or equivalently, only 0 and -1 are present in the "list notation". Here are some identities we found that also allow 1 (again, a very incomplete list):

(11) {[0,1,0]}, [{[0]}, {}, {}], {} → [-1, -1, -1, -1, -1, 1, -1, -1, -1, -1, -1, 0] (mod 12)

(12) {}, [{[1], [2]}, {[2]}, {[0], [3]}], {} → [-1, -1, -1, 1, 0, -1, -1, -1, -1, 0, 0, -1] (mod 12)

(13) {}, [{[1], [2]}, {[0], [2], [3]}, {[0], [3]}], {} → [-1, 0, -1, 1, 0, -1, -1, -1, -1, 0, 0, -1] (mod 12)

(14) {[1, 2], [2, 1]}, [{}, {[0], [1], [2], [3]}, {[2]}], {[1], [2]} → [-1, 0, -1, 1, 0, -1, -1, -1, -1, 0, 0, -1] (mod 12)

Future work

1. Search for larger modulo identities. We have already tried this out for a small batch of inputs, and one identity we found is the following: {[0]}, [{[2], [1, 1]}, {[1, 2], [3, 2]}], {} → [-1, 0, -1, 0, -1, 1, -1, -1, -1, 1, -1, -1, -1, 1, -1, 0, -1, 0, -1, 0] (mod 20). We are hopeful that we will find many more such identities.

2. Put more variations on the initial conditions.

3. Currently our approach only deals with conditions on contiguous sub-partitions. It will be nice to develop a general frame work/an efficient way to search for identities that avoid sub-partitions that are not necessarily contiguous (like in the infinite family we presented).

Drew Sills and Ali Uncu's suggestions

1. Some identities have "wierd" "sum side", for example, the big Göllnitz companion identity "sum side" requires difference of at least 6 between parts EXCEPT that it is ok if the smallest two parts are 1 and 6. Maybe many such "wierd" partition identites are out there, we would like to search for them.

2. Incorporate Nandi's $*$ operator, which is the asterisk in the pattern $[3, 2*, 3, 0]$ into our program to search for more Nandi-type partition identities. It is not hard to adapt our algorithm to look for Nandi-type partitions, in fact, we have already done that. But we would need more insight on "where to look", as some initial searches did not help us find new identities.

Thank you!

Questions?