An Exploration of Nested Recurrences Using Experimental Mathematics

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Outline

1. Nested Recurrences
   - Slow Solutions
   - Linear-Recurrent Solutions

2. Discovering More Golomb/Ruskey-Like Solutions

3. Special Initial Conditions
   - 1 through N
   - Other Initial Conditions

Recurrence Relations

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- The terms in a solution that don't satisfy the recurrence are called the initial condition.

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- Closed forms for solutions, rational generating functions

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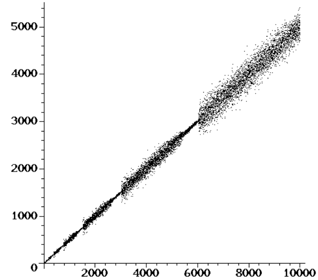
  - Many open questions of the form "Does this sequence even exist?"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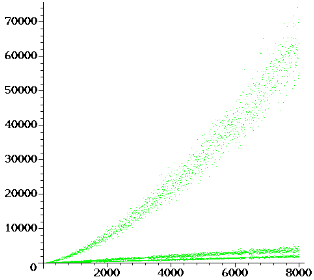
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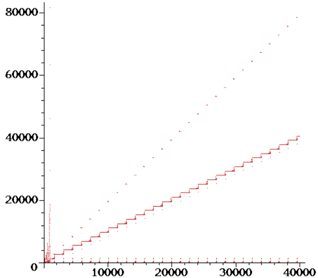
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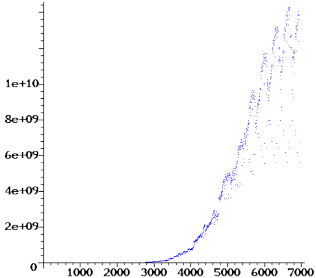
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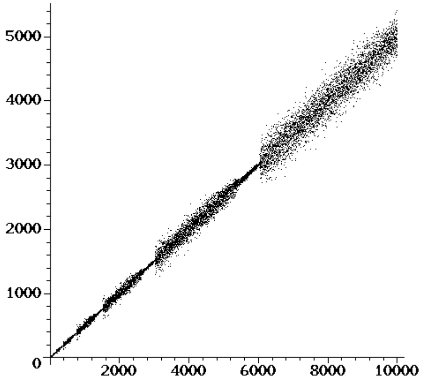
The Hofstadter Q-Sequence

Plot of First 10000 Terms



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- Open Question: Does the Hofstadter Q-sequence die?

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- Not really cheating:
  Can still ask: "Does $Q(n-1)$ ever exceed n?"

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Often solutions to other related recurrences

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   - Linear-Recurrent Solutions

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- Some, like Conolly's sequence, have combinatorial interpretations in terms of counting leaves in certain tree structures.
- Others have no known such interpretations.

Slow Solutions

Other Slow Solutions to Nested Recurrences

- Hofstadter-Conway \$10000 Sequence (A004001):
  $A(n) = A(A(n-1)) + A(n - A(n-1))$,
  I.C. $\langle 1, 1 \rangle$ [Conway, Mallows]

  1, 1, 2, 2, 3, 4, 4, 4, 5, 6, 7, 7, 8, 8, 8, 8, 9, 10, 11, 12, 12

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- Hofstadter V-sequence (A063882):
  $V(n) = V(n - V(n-1)) + V(n - V(n-4))$,
  I.C. $\langle 1, 1, 1, 1 \rangle$ [Balamohan, Kuznetsov, Tanny]

  1, 1, 1, 1, 2, 3, 4, 5, 5, 6, 6, 7, 8, 8, 9, 9, 10, 11, 11, 11, 12

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- $B(n) = B(n - B(n-1)) + B(n - B(n-2)) + B(n - B(n-3))$,
  I.C. $\langle 1, 2, 3, 4, 5 \rangle$ [F., A278055]

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Golomb's Solution

Golomb's Sequence (1990)

- Same recurrence as Hofstadter:
  $Q_G(n) = Q_G(n - Q_G(n-1)) + Q_G(n - Q_G(n-2))$

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First few terms (A244477):
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Formula

- $Q_G(3k) = 3k - 2$
- $Q_G(3k+1) = 3$
- $Q_G(3k+2) = 3k + 2$

Proof of Golomb's Solution

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- $Q_G(1) = 3$
- $Q_G(2) = 2$
- $Q_G(3) = 1$

Proof.

Proof by induction

□

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$$\begin{aligned} Q_G(3k) &= Q_G(3k - Q_G(3k-1)) + Q_G(3k - Q_G(3k-2)) \\ &= Q_G(3k - Q_G(3(k-1)+2)) + Q_G(3k - Q_G(3(k-1)+1)) \end{aligned}$$

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- $Q_G(3k) = 3k - 2$
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Proof by induction

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Base case: Initial conditions □

Ruskey's Solution

Ruskey's Sequence (2011)

- Same recurrence as Hofstadter:
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First few terms (A188670):
3, 6, 5, 3, 6, 8, 3, 6, 13, 3, 6, 21, 3, 6, 34, 3, 6, 55, 3, 6, 89, 3, 6, 144, 3, 6, 233, 3, 6, 377, 3, 6, 610, 3, 6, 987, 3, 6, 1597

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Formula

- $Q_R(3k) = F(k+4)$, where F means Fibonacci
- $Q_R(3k+1) = 3$
- $Q_R(3k+2) = 6$

1. Nested Recurrences
   - Slow Solutions
   - Linear-Recurrent Solutions

2. Discovering More Golomb/Ruskey-Like Solutions

3. Special Initial Conditions
   - 1 through N
   - Other Initial Conditions

General Framework

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Steps (to be illustrated by example)

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6. Try to satisfy the constraints
7. Find an initial condition

First Two Steps

Running Example

We'll discover another solution to the Q-recurrence with 3 interleaved subsequences.

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Unpacking the Recurrence Inductively

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Structural Consistency

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Running Example

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  - Need $3k + \mu_0 = 3k + \mu_0$, so $0 = 0$ (tautology)

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$1, 0, 3, 3, 2, 6, 3, 2, 9, 3, 2, 12, 3, 2, 15, 3, 2, 18, 3, 2, 21, 3, 2, 24, 3, 2, 27, \ldots$ (A264756)

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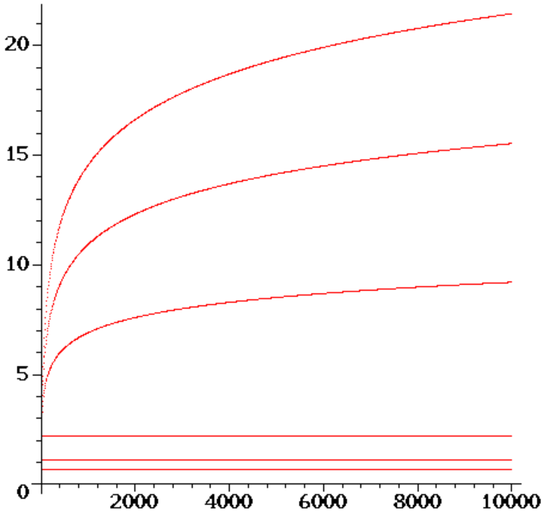
  - For Q, can find a degree d polynomial if $m = 3d$



Sample solution, log plot, $m = 9$, cubic subsequence (A264758)

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1. Nested Recurrences
   - Slow Solutions
   - Linear-Recurrent Solutions

2. Discovering More Golomb/Ruskey-Like Solutions

3. Special Initial Conditions
   - 1 through N
   - Other Initial Conditions

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- Can consider weak or strong death

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  - New initial condition: Old sequence through the end of the last pattern

Q-Recurrence

Primary exploration: Q-recurrence with I.C. $\langle 1, 2, 3, \ldots, N\rangle$

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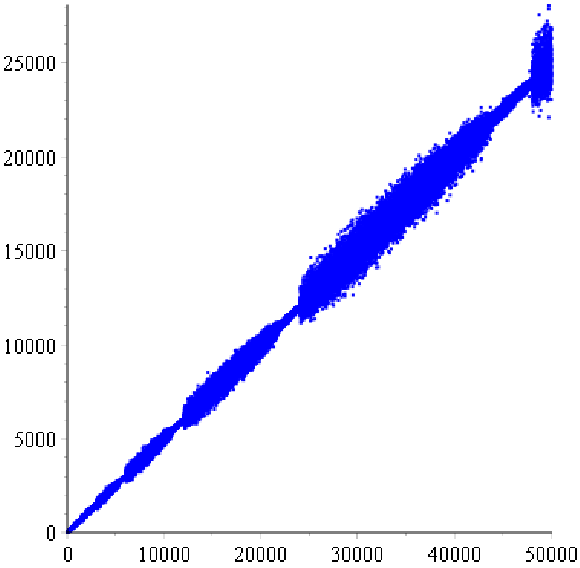
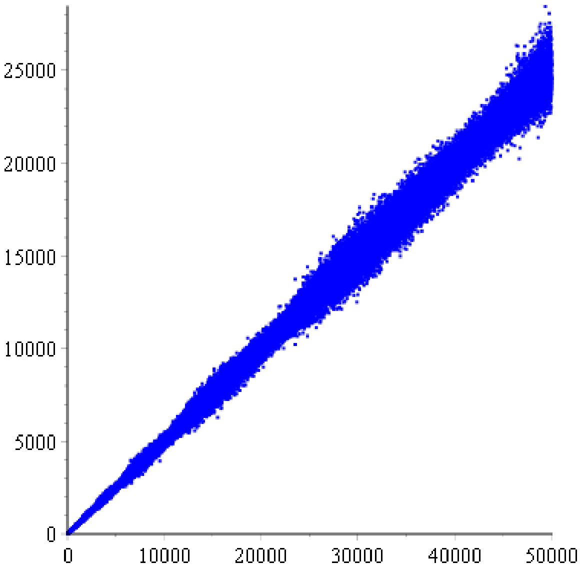
- $N = 4, 5, 6, 7, 9, 10, 13$ each persist for at least 30 million terms

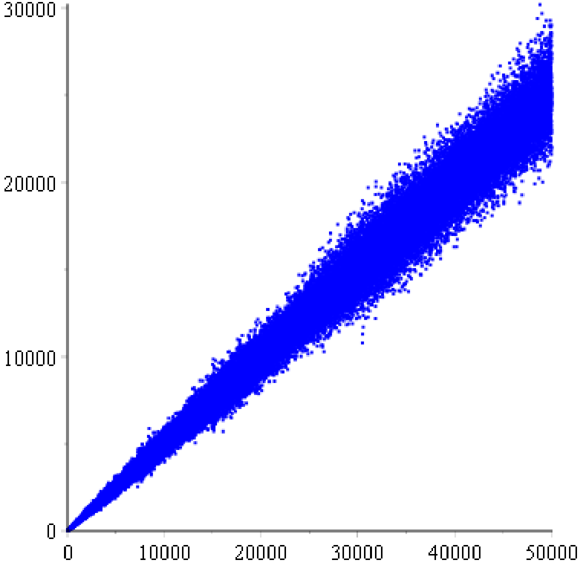


$N = 3$ (Q_3, A005185)

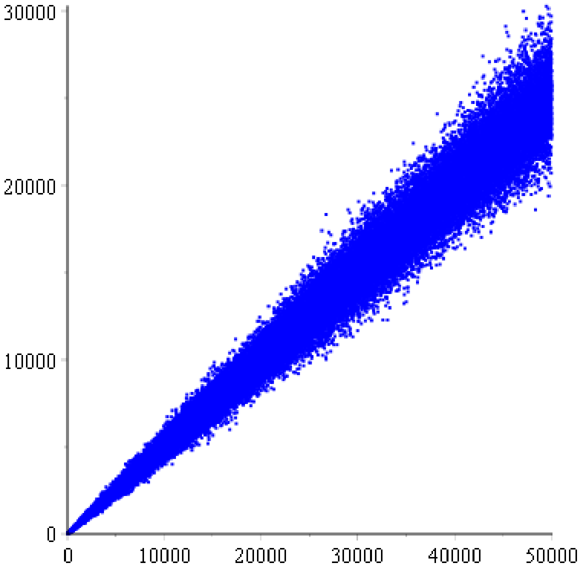


25000
20000
15000
10000
5000
0
0
10000
20000
30000
40000
50000

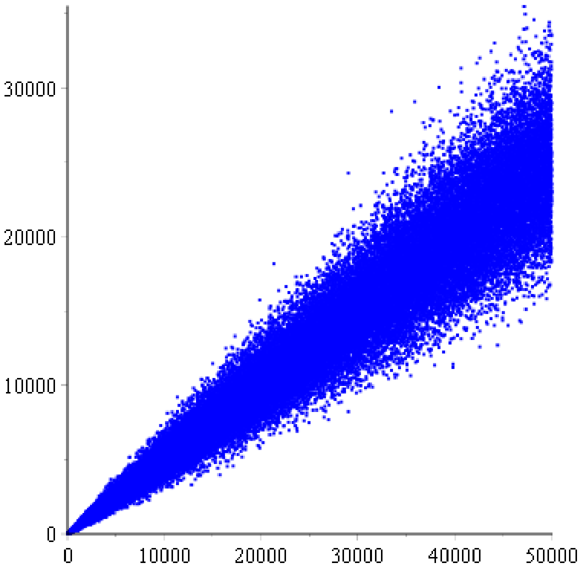
Q_4, A278056



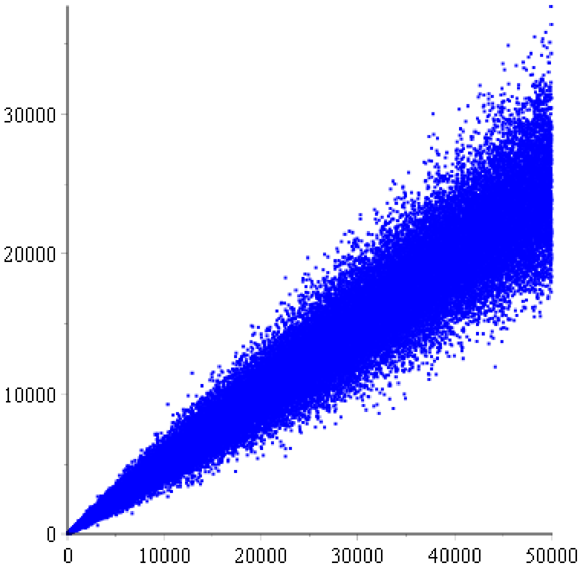
Q_5, A278057



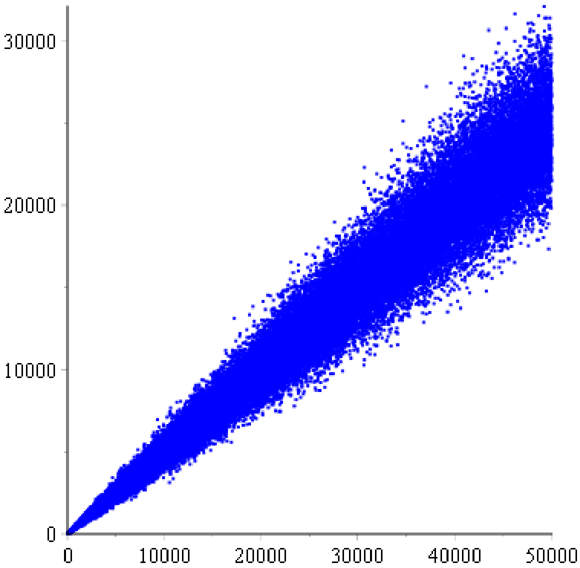
Q_6, A278058



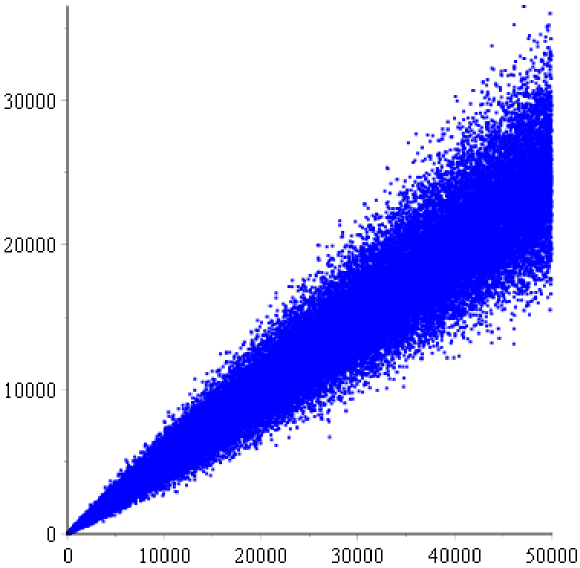
Q_7, A278059



Q_9, A278061



Q_{10}, A278062



Q_{13}, A278065

Q-Recurrence: Weak Death

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For all $N \geq 14$, Q_N weakly dies.

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$$\begin{aligned} Q_N(N+1) &= Q_N(N+1-Q(N)) + Q_N(N+1-Q(N-1)) \\ &= Q_N(N+1-N) + Q_N(N+1-(N-1)) \\ &= Q_N(1) + Q_N(2) \\ &= 1+2=3 \end{aligned}$$

Initial Condition 1 through N: Weak Death

Proof.

- $Q_N(N+1) = 3$
- $Q_N(N+2) = N+1$
- $Q_N(N+3) = N+2$
- $Q_N(N+4) = 5$
- $Q_N(N+5) = N+3$
- $Q_N(N+6) = 6$
- $Q_N(N+7) = 7$
- $Q_N(N+8) = N+4$
- $Q_N(N+9) = N+6$
- $Q_N(N+10) = 10$
- $Q_N(N+11) = 8$
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- $Q_N(N+18) = 11$
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- $Q_N(N+22) = 13$
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Check 14, 15, 16, 17, 18, 19, 20 separately. They all weakly die. □

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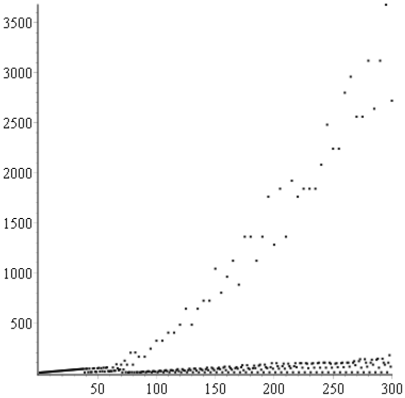
  - $N \equiv 4 \pmod 5$: Strong death after $2N + 8$ terms

$N \equiv 3 \pmod 5$ is Weird



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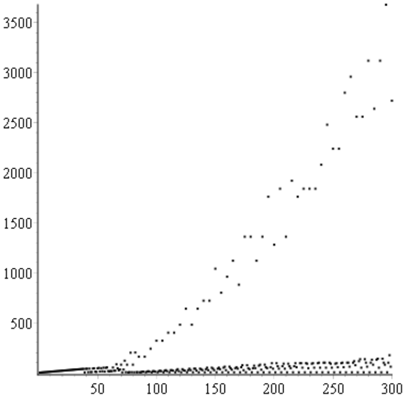

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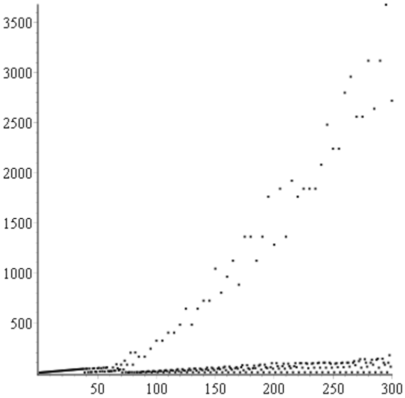

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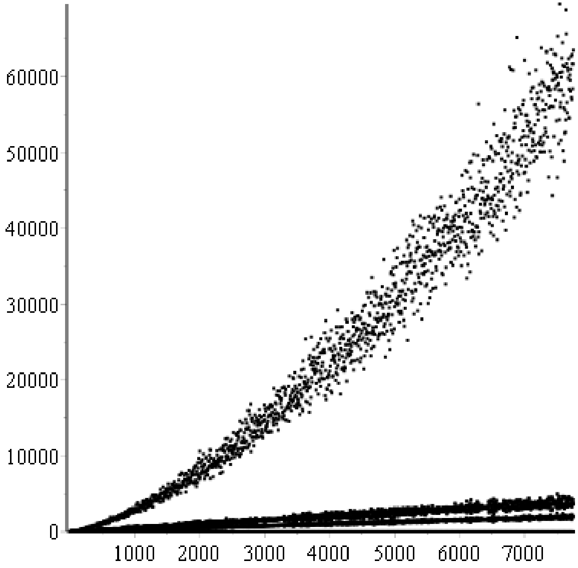
Rest of terms are poorly understood



Another solution isolating these terms, A272610, Initial Condition $\langle 5, 9, 4, 6\rangle$

$N \equiv 2 \pmod{5}$ is Even Weirder

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- Can continue depending on N mod higher powers of 5

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- After this, value of $(A_{i+1} + 2i + 3) \bmod 5$ determines next behavior

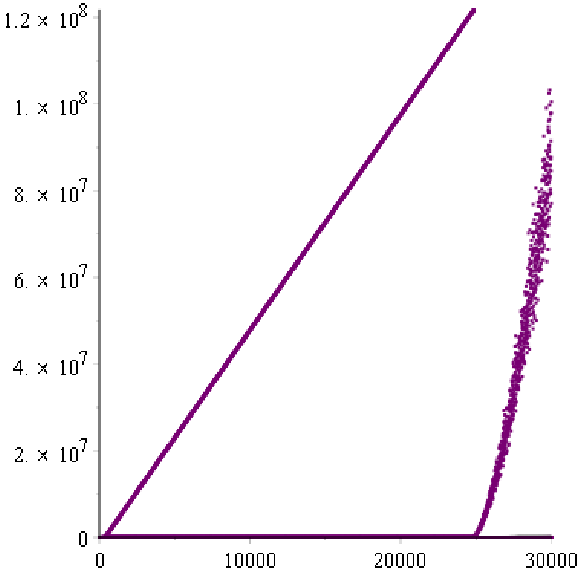
  0: Strong death after 160 more terms (like 1 mod 5)

  1: Keep going with $i + 1$ (like 2 mod 5)

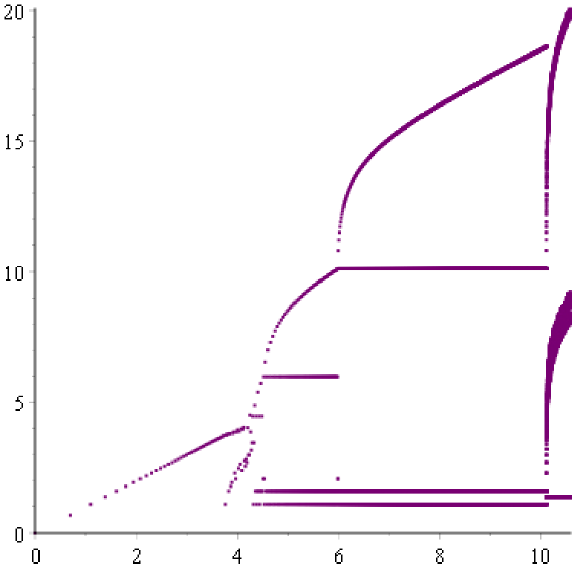
  2: Fours and chaos forever (like 3 mod 5)

  3: Strong death after 4 more terms (like 4 mod 5)

  4: Strong death after 14 more terms (like 0 mod 5)



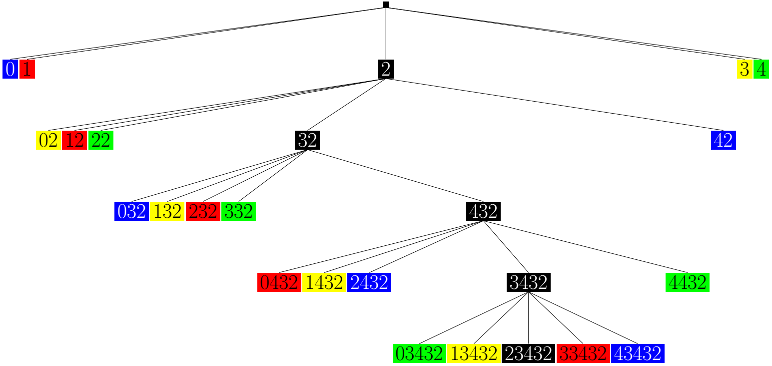
Q_{42}, A274055



Q_{42}, both axes log scale, A274055

Tree of Behaviors of Q_N

Write N in base 5, read digits from right to left



Death 160   Go Deeper   Fours and Chaos   Death 4   Death 14

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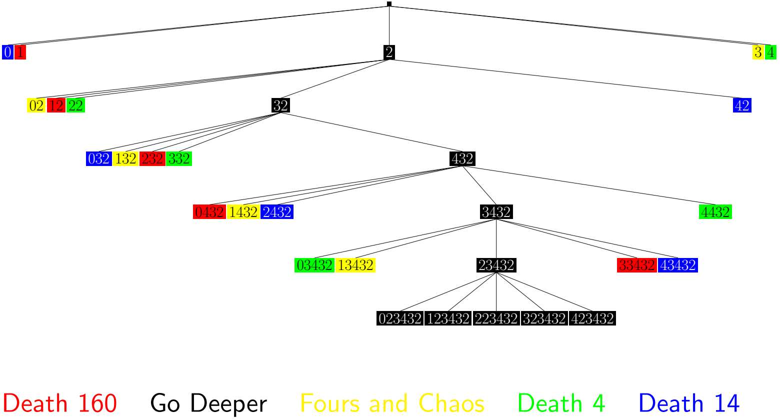

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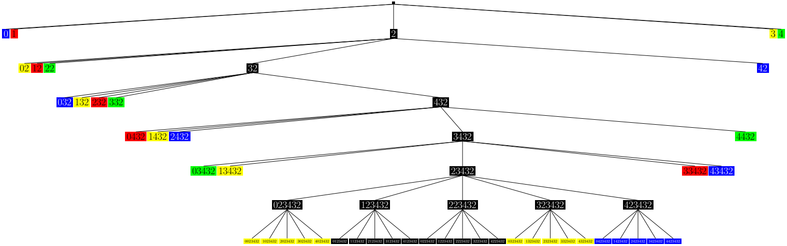

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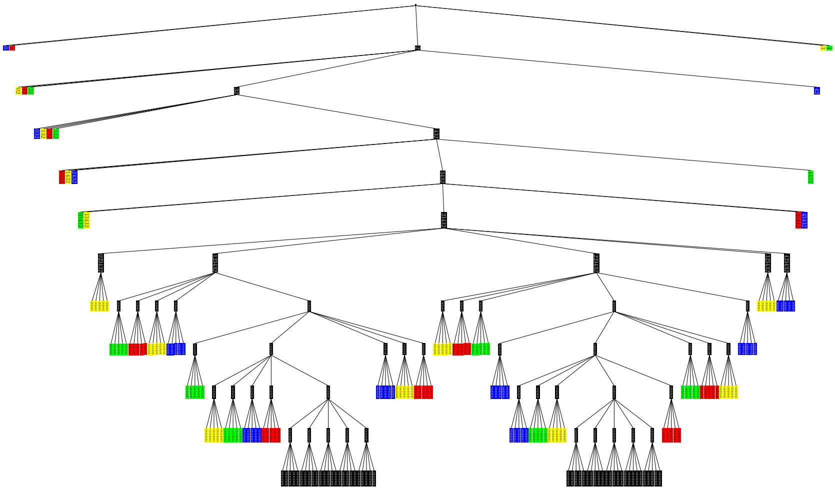

Three-Term Hofstadter-like Recurrence

$B_N(n) = B_N(n - B_N(n-1)) + B_N(n - B_N(n-2)) + B_N(n - B_N(n-3))$, initial condition $\langle 1, 2, 3, \ldots, N \rangle$

Structure Theorem for B_N

- $N \geq 74$: B_N does not strongly die before $2N$ terms; has period-7 quasilinear pattern from $B_N(N+67)$ through roughly $B_N(2N)$.

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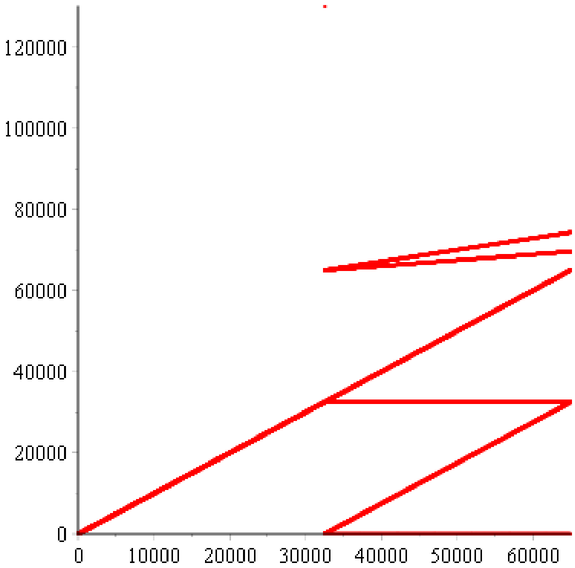
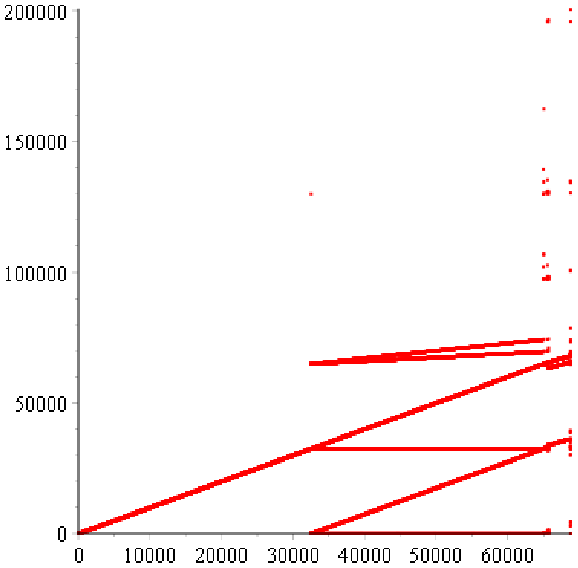
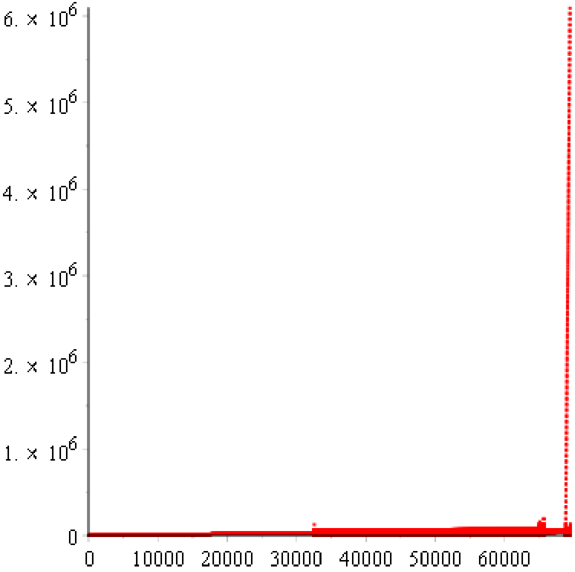


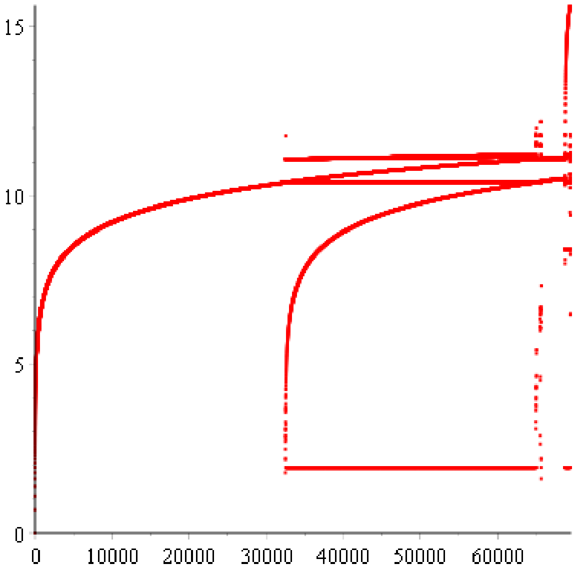
First 64964 terms of B_{32478}, A274058



First 68814 terms of B_{32478}, A274058



All 69503 terms of B_{32478}, A274058



All 69503 terms of B_{32478}, log plot, A274058

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- Fun fact: B_{20830} strongly dies, but it has $84975 \cdot 2^{560362} + 31$ terms.

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  - Built out of infinitely many period-5 sub-patterns

More on Sporadic N Values

Facts

- $N \in \{81, 182, 429, 822, 1892, 2789, 7292, 23511, 25163\}$:
  - Eventual alternation between 2 and $M \cdot 2^k$ for some M
  - So, doesn't strongly die for a "boring" reason
- $N \in \{193, 3442\}$:
  - Built out of infinitely many period-5 sub-patterns
  - Each one six times longer than previous

More on Sporadic N Values

Facts

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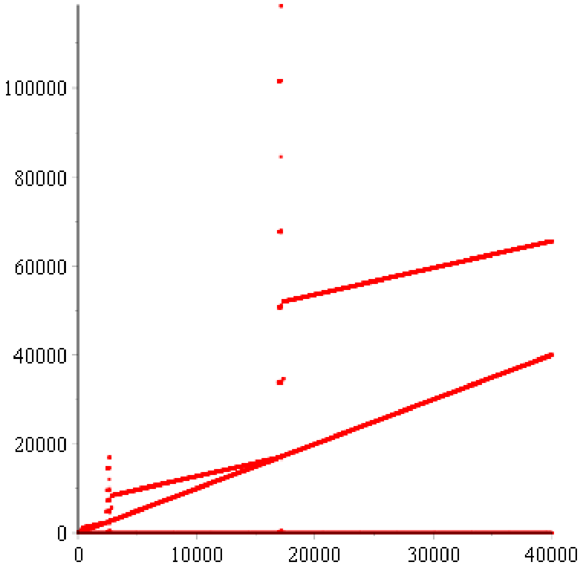
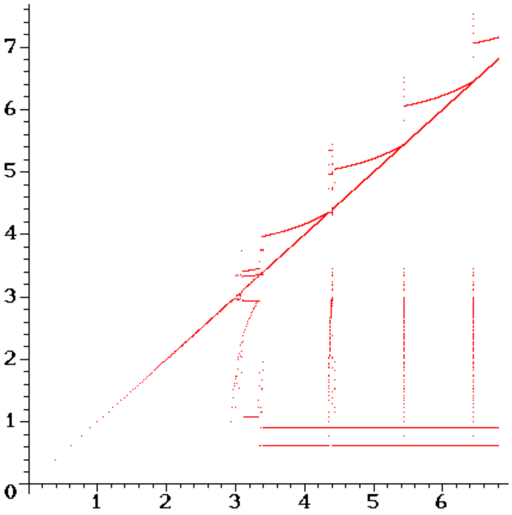
- $N \in \{193, 3442\}$:
  - Built out of infinitely many period-5 sub-patterns
  - Each one six times longer than previous
  - So, doesn't strongly die for an "interesting" reason



First 40000 terms of B_{193} (A283884)



First 200000 terms of B_{193}, both axes log (A283884)

Four-Plus-Term Hofstadter-like Recurrence

$$G_{d,N}(n) = \sum_{i=1}^{d} G_{d,N}(n - G_{d,N}(n - i))$$

Initial condition $\langle 1, 2, 3, \ldots, N \rangle$

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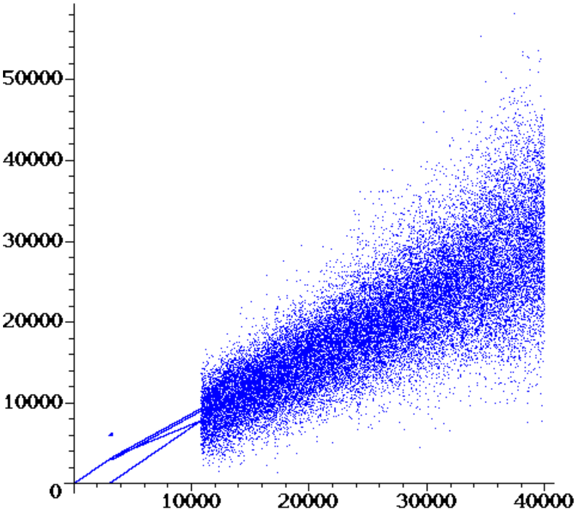
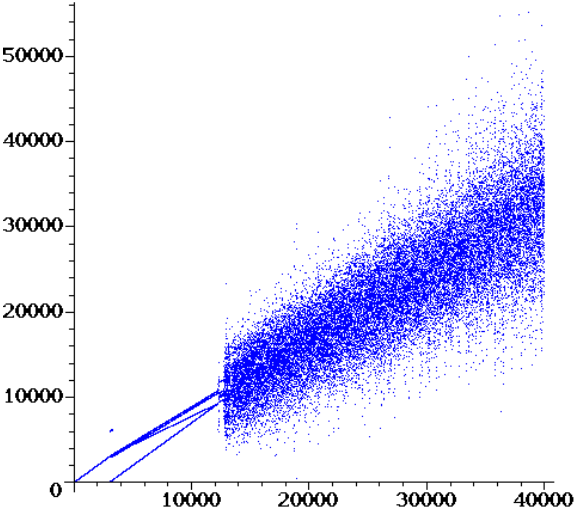
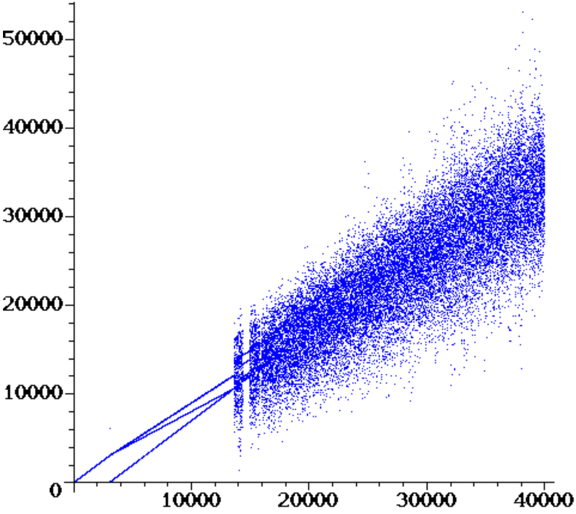
Really weird behavior; see for yourself!



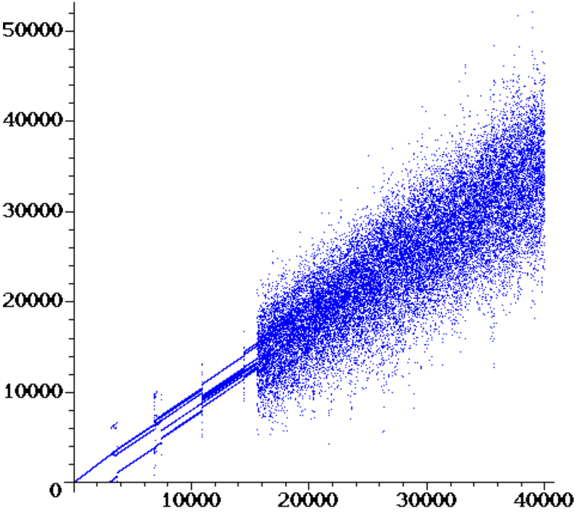
First 40000 terms of $G_{4,3000}$



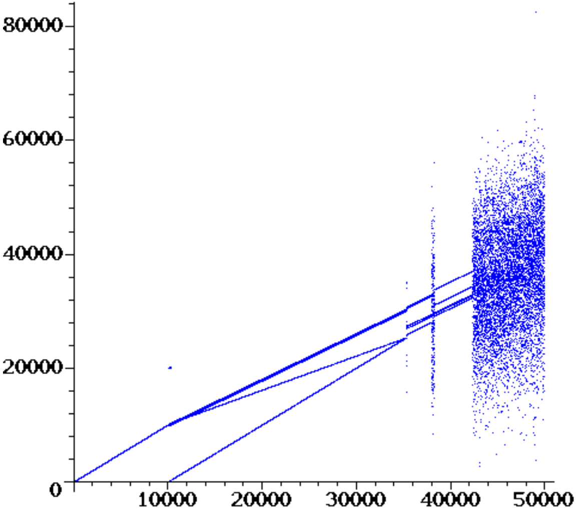
First 40000 terms of $G_{5,3000}$



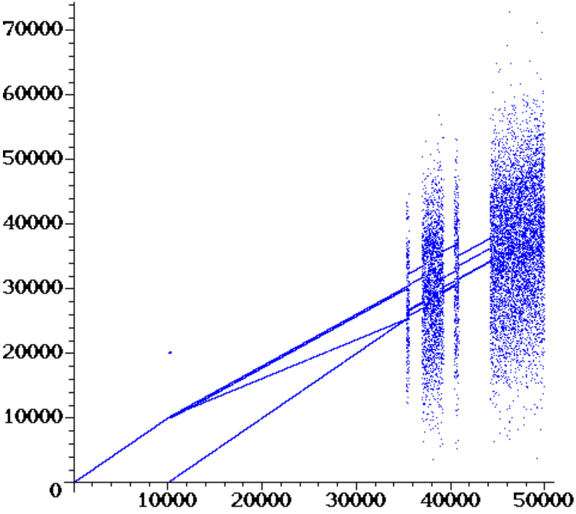
First 40000 terms of $G_{6,3000}$



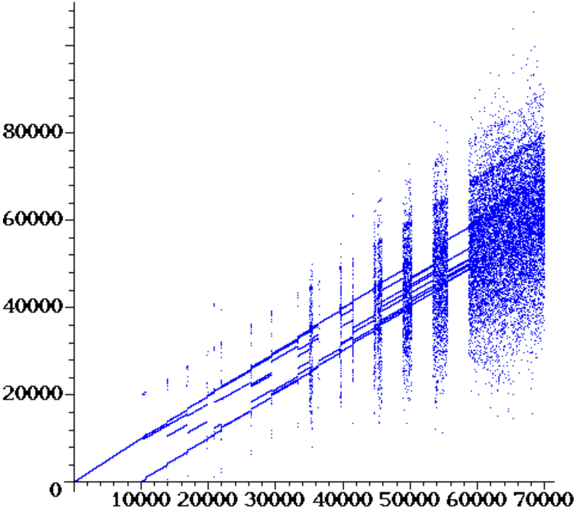
First 40000 terms of $G_{7,3000}$



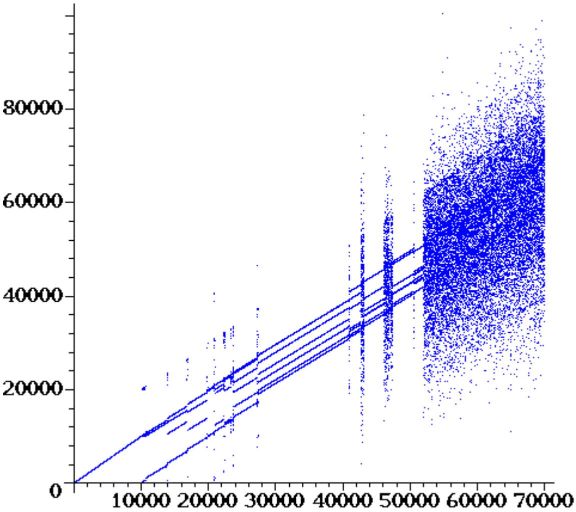
First 50000 terms of $G_{4,10000}$ (A283889)



First 50000 terms of $G_{4,10001}$ (A283890)



First 70000 terms of $G_{7,10000}$ (A283891)



First 70000 terms of $G_{7,10001}$ (A283892)

Other Interesting Initial Conditions

We Consider Q-Recurrence With:

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Pretty much any other parametrized family of initial conditions that you can think of is worth exploring!

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Pretty much any other parametrized family of initial conditions that you can think of is worth exploring!

Can also do all these same explorations with other recurrences

$\langle N, 2\rangle$ and $\langle 2, N\rangle$

Facts

- Most sequences quasilinear and easy to describe

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- $\langle N, 2\rangle$, $N \geq 25$, $N \equiv 3 \pmod 4$: Strong death after $5N + 11$ terms
- $\langle N, 2\rangle$, $N \geq 75$, $N \equiv 5 \pmod{12}$: Strong death after $28N + 64$ terms

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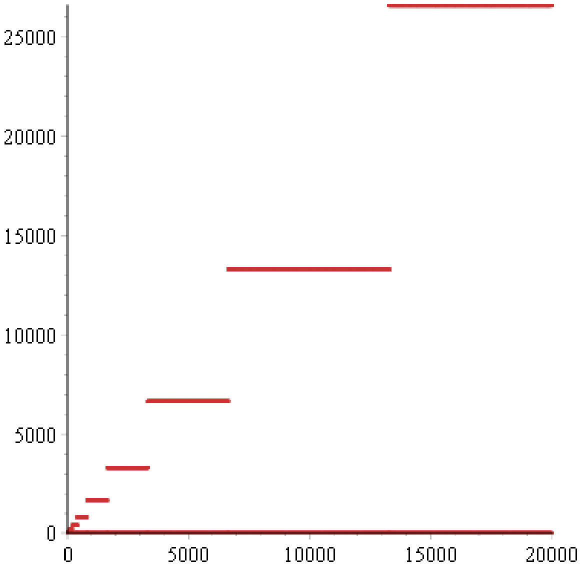
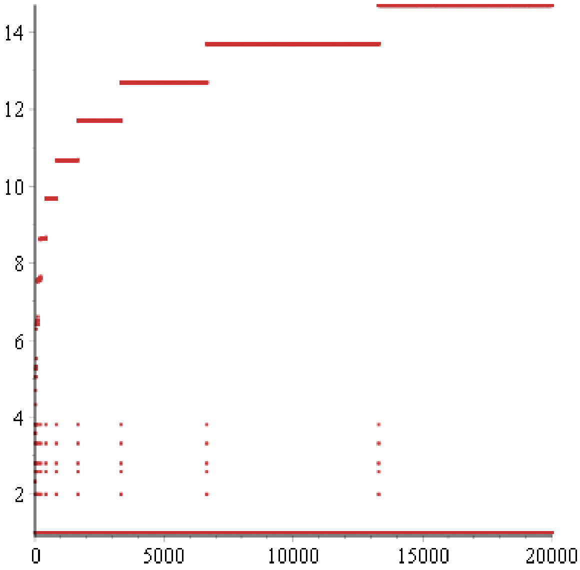
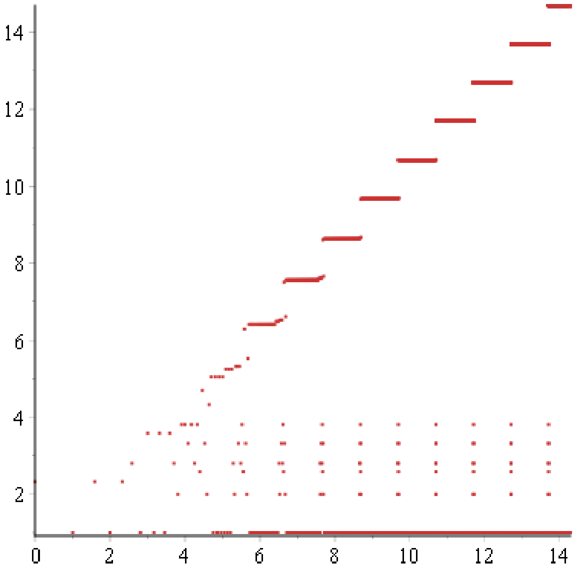
- $\langle N, 2\rangle$: A few sporadic interesting cases for small N
  - Most notably $N = 5$, $N = 17$, $N = 41$



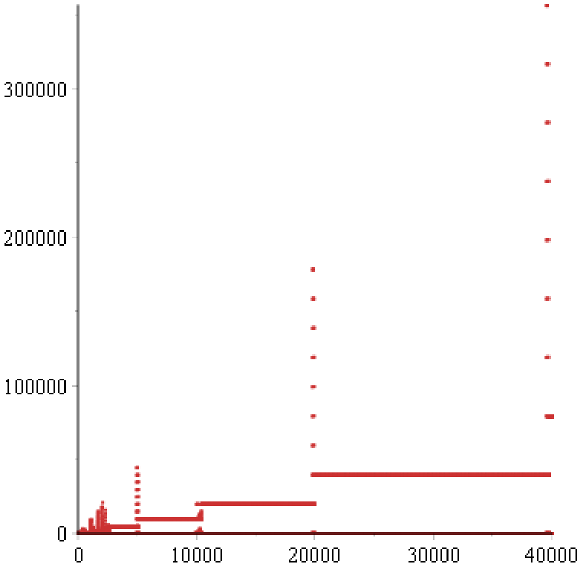
Initial condition $\langle 5, 2 \rangle$, A278066



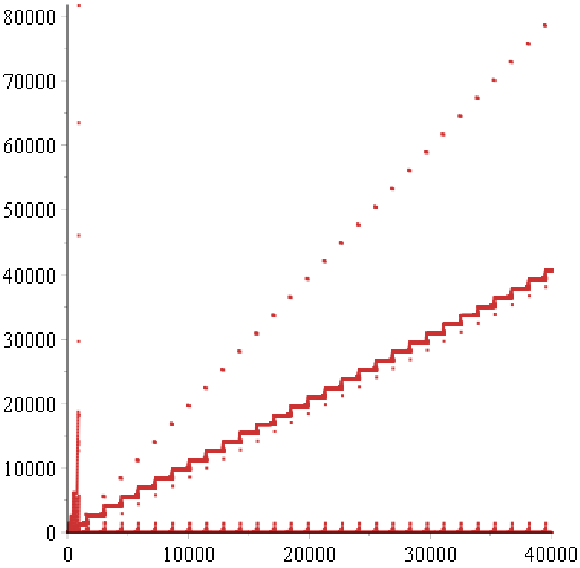
Initial condition $\langle 5, 2 \rangle$, log plot, A278066



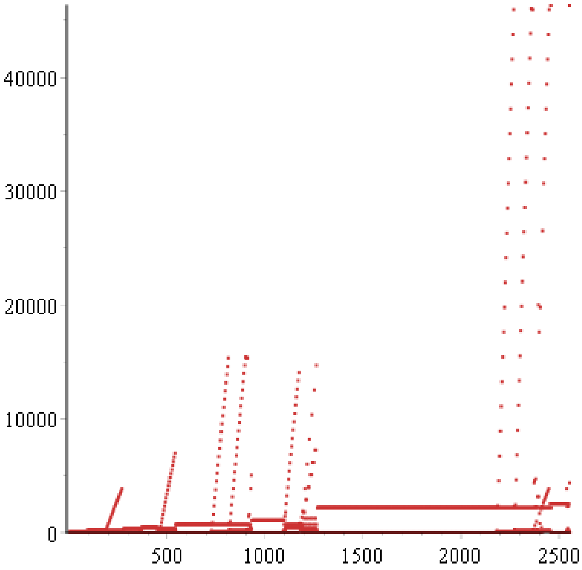
Initial condition $\langle 5, 2 \rangle$, log-log plot, A278066



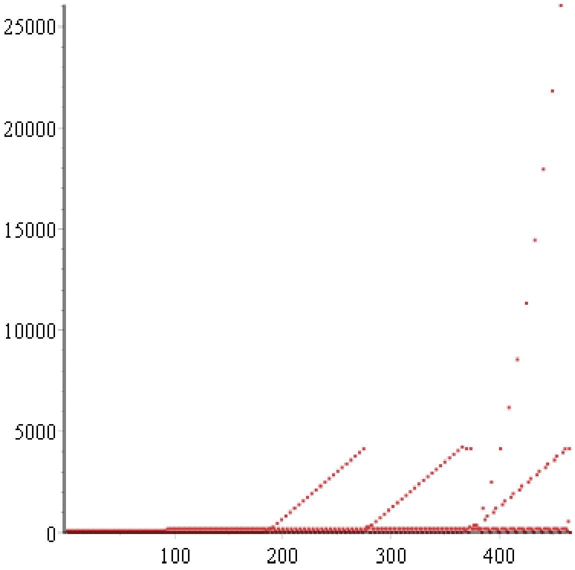
Initial condition $\langle 41, 2\rangle$



Initial condition $\langle 57, 2\rangle$, A278068



Initial condition $\langle 89, 2 \rangle$, A283896



Initial condition $\langle 91, 2\rangle$, A283897

$\langle N, 4, N, 4 \rangle$

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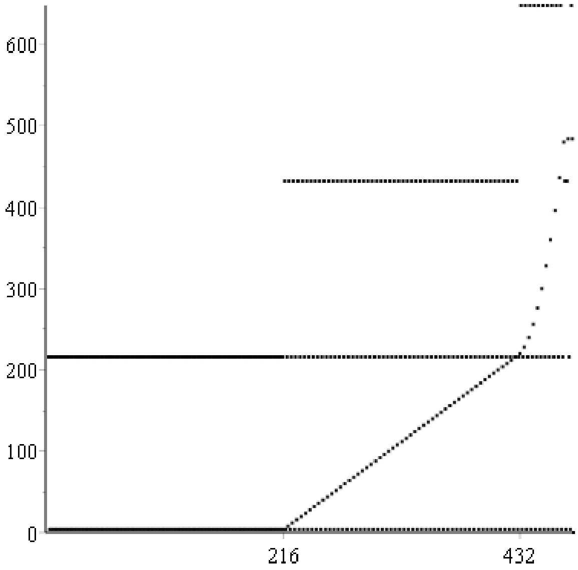
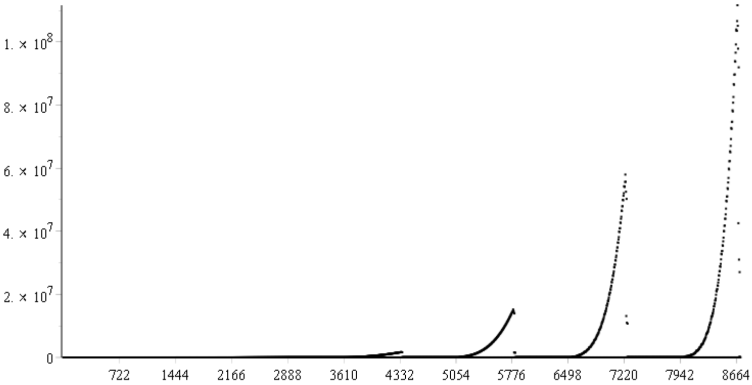
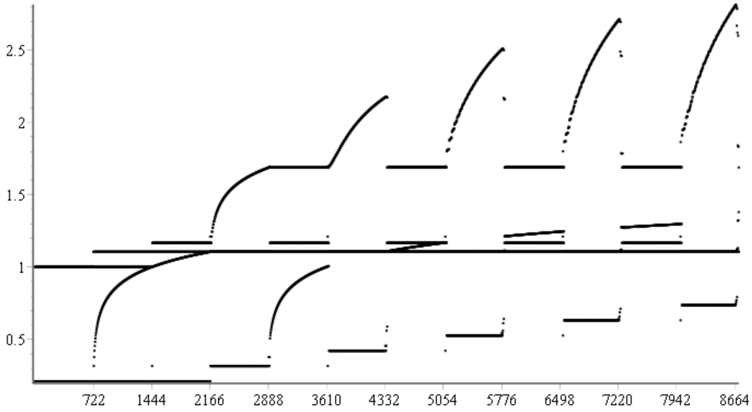
- $N = 2A^2 + 2A$: Seems to strongly die eventually, but complicated



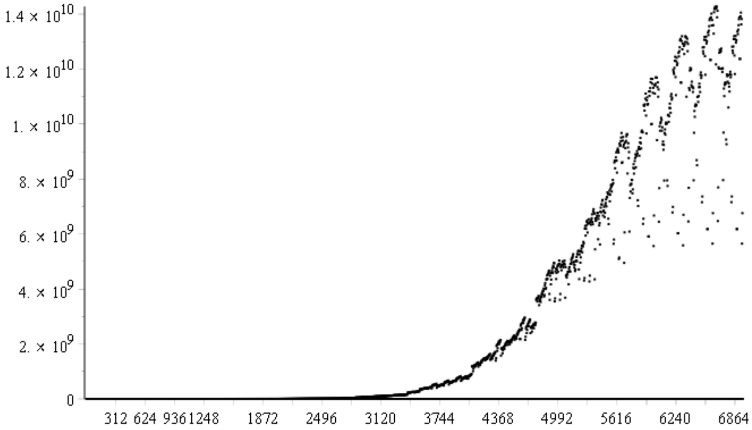
$\langle 216, 4, 216, 4 \rangle$, all 481 terms (similar to A283899)



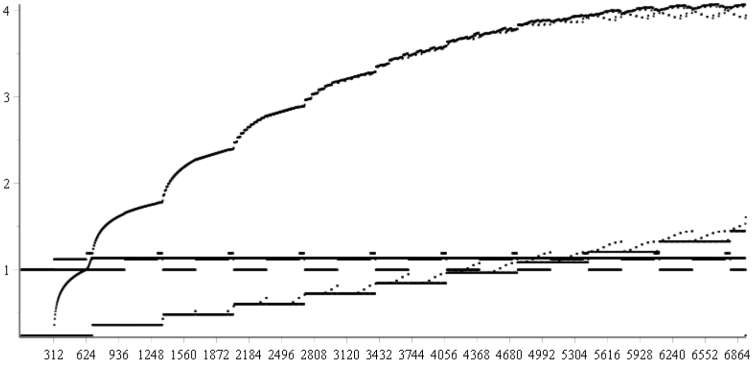
$\langle 722, 4, 722, 4 \rangle$, all 8714 terms (similar to A283900)



$\langle 722, 4, 722, 4 \rangle$, all 8714 terms, log plot (similar to A283900)



$\langle 312, 4, 312, 4 \rangle$, all 6944 terms (A283898)



312, 4, 312, 4, all 6944 terms, log plot (A283898)

$\langle 4, N, 4, N \rangle$

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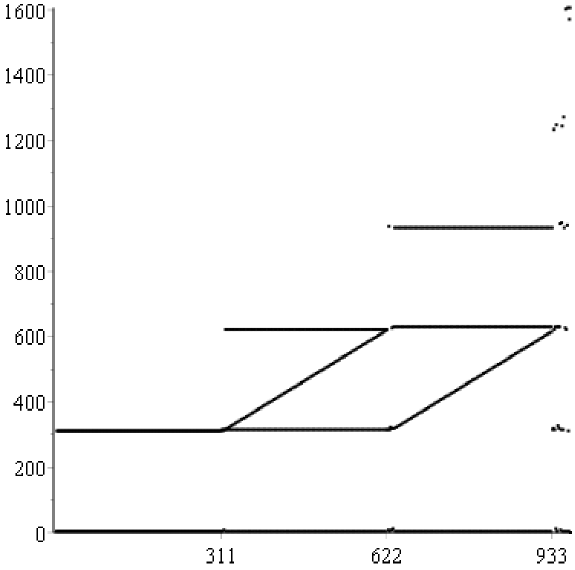
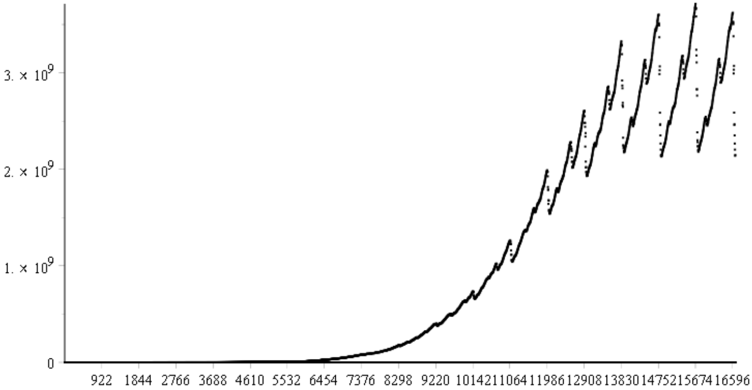
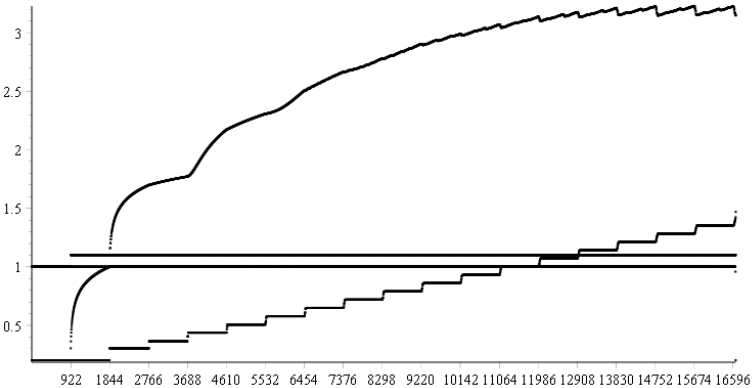


$\langle 4, 311, 4, 311 \rangle$, all 969 terms (A283901)



$\langle 4, 922, 4, 922 \rangle$, all 16667 terms (similar to A283902)



$\langle 4, 922, 4, 922 \rangle$, all 16667 terms, log plot (similar to A283902)

Summary

We've seen a huge diversity of solutions to nested recurrences

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My mantra when working with nested recurrences: "If you think it might be possible, it probably is possible."

References I

[1] B. Balamohan, A. Kuznetsov, and Stephen Tanny, *On the behavior of a variant of Hofstadter's Q-sequence*, J. Integer Seq. **10** (2007), 29.

[2] B. W. Conolly, *Meta-Fibonacci sequences*, *Fibonacci & Lucas Numbers, and the Golden Section*, 1989, pp. 127–138.

[3] Nathan Fox, *Finding linear-recurrent solutions to Hofstadter-like recurrences using symbolic computation*, arXiv preprint arXiv:1609.06342 (2016).

[4] ———, *Linear recurrent subsequences of generalized meta-Fibonacci sequences*, J. Difference Equ. Appl. (2016).

[5] ———, *Quasipolynomial solutions to the Hofstadter Q-recurrence*, Integers **16** (2016), A68.

[6] ———, *A slow relative of Hofstadter's Q-sequence*, arXiv preprint arXiv:1611.08244 (2016).

[7] S.W. Golomb, *Discrete chaos: Sequences satisfying "Strange" recursions* (1991).

[8] Douglas Hofstadter, *Gödel, Escher, Bach: an Eternal Golden Braid*, Basic Books, New York, 1979.

[9] Abraham Isgur, David Reiss, and Stephen Tanny, *Trees and meta-Fibonacci sequences*, Electron. J. Combin. **16** (2009), no. R129, 1.

[10] Colin L Mallows, *Conway's challenge sequence*, Amer. Math. Monthly **98** (1991), no. 1, 5–20.

References II

[11] F. Ruskey, *Fibonacci meets Hofstadter*, Fibonacci Quart. **49** (2011), no. 3, 227–230.

[12] N.J.A. Sloane, *OEIS Foundation Inc.*, The On-Line Encyclopedia of Integer Sequences, 2016. http://oeis.org/.

[13] Stephen M Tanny, *A well-behaved cousin of the Hofstadter sequence*, Discrete Math. **105** (1992), no. 1, 227–239.

Thank you!

yep