

**EXPERIMENTAL MATHEMATICS APPLIED TO THE
STUDY OF NON-LINEAR RECURRENCES**

by

EMILIE ANN HOGAN

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ABSTRACT OF THE DISSERTATION

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By EMILIE ANN HOGAN

Dissertation Director: Doron Zeilberger

In this thesis we study three topics within the broad field of nonlinear recurrences. First we will consider global asymptotic stability in rational recurrences. A recurrence is globally asymptotically stable when the sequence it produces converges to an equilibrium solution given any initial conditions. Up to now, this topic has not been studied from an algorithmic perspective. We develop an algorithm that takes as input a rational recurrence relation conjectured to be globally asymptotically stable, and, if it is, outputs a *rigorous proof* of its stability. We apply this algorithm to many specific rational recurrences.

Secondly, we study a three-parameter family of rational recurrences that produce sequences of integers. We apply two methods to prove the integrality of these sequences. We first show that some of the sequences also satisfy a linear recurrence. In order to establish integrality of the entire family we make use of the Laurent phenomenon [11].

Finally, we develop a new concept that generalizes the notion of a recurrence. Instead of producing a single sequence, we produce infinitely many sequences from one set of initial conditions. We will study two families of this type of generalized recurrences that produce rational numbers when complex numbers are expected. We also observe exponential sequences being produced by some of these generalized recurrences.

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Dedication

To my friends and family:
thank you for always believing that I could achieve this goal.

Especially to my grandpa, David Dye,
whose passion for mathematics I strive to replicate.

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Chapter 1

Introduction

In this thesis I study nonlinear recurrences by applying computerized and experimental methods. We will see computers being utilized in two different ways. First, in Chapter 2, we study a general algorithm developed to prove convergence of rational difference equations. Here we utilize computers as proof machines. The algorithm is created by human means, programmed in Maple, and then used to systematically produce convergence theorems. In contrast, in Chapters 3 and 4 experimental methods are used to form conjectures. In Chapter 3, we study nonlinear recurrences that produce sequences which also satisfy a linear recurrence. These linear recurrences could not have been conjectured without the use of experimental mathematics. Additionally, the theorems and propositions contained in Chapter 4 were all conjectured by noticing interesting patterns that arose when iterating a more general type of recurrence.

Before continuing I must first lay the foundation for the study of recurrences. Broadly speaking, a recurrence is a function, $f : \mathbb{N} \rightarrow \mathbb{C}$, defined in terms of itself. The most common example is the Fibonacci recurrence where

$$f_n = f_{n-1} + f_{n-2}, \quad f_1 = f_2 = 1.$$

We can also think of a recurrence as being described by a function $F : \mathbb{N} \times \mathbb{C}^k \rightarrow \mathbb{C}$ (see p. 8 in [8]):

$$f_n = F(n, f_{n-1}, \dots, f_{n-k}).$$

In order to create a sequence from the recurrence we must also be given a set of k initial values $\{x_1, \dots, x_k\} \subset \mathbb{C}$, so that $f_1 = x_1, \dots, f_k = x_k$. In the case of the Fibonacci recurrence, $F(n, x, y) = x + y$, and $x_1 = x_2 = 1$. We call the integer k the *order* of the recurrence. Notice that in the Fibonacci recurrence, the function F does not

depend on n . When this happens we say that the recurrence has *constant coefficients*. All of the recurrences we will be considering will have constant coefficients. Therefore, throughout the thesis we will drop the dependence of F on n .

The class of all recurrences can be decomposed into two sub-classes: linear and nonlinear. Linear recurrences are those for which F is a linear function. The simplest non-trivial example is the Fibonacci recurrence. Of course, the recurrences defined by $F(x) = cx$ and $F = c$ for some constant $c \in \mathbb{C}$ are simpler, but quite trivial. In the former, it is easily seen that $f_n = x_1 c^n$, and in the latter $f_n = c$ for all $n \in \mathbb{N}$. Within the class of linear recurrences we can further distinguish homogeneous recurrences, those for which the constant term in F is 0, and non-homogeneous recurrences. The following theorem, which gives us a closed form formula for the n^{th} term in the sequence produced by a linear homogeneous recurrence, appears in Chapter 4 of Richard Stanley's book *Enumerative Combinatorics: Volume I* [34].

Theorem 1.0.1. *Let $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{C}$ be fixed, $k \geq 1$ and $\alpha_k \neq 0$. The following conditions on a function $f : \mathbb{N} \rightarrow \mathbb{C}$ are equivalent:*

(i) For all $n \geq 0$,

$$f_{n+k} + \alpha_1 f_{n+k-1} + \alpha_2 f_{n+k-2} + \dots + \alpha_k f_n = 0.$$

(ii) For all $n \geq 0$,

$$f_n = \sum_{i=1}^m P_i(n) \gamma_i^n,$$

where $1 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_k x^k = \prod_{i=1}^m (1 - \gamma_i x)^{d_i}$, the γ_i s are distinct, and $P_i(n)$ is a polynomial in n of degree less than d_i .

So we see that the class of linear homogeneous recurrences is completely characterized. Non-homogeneous linear recurrences are not quite so clear cut, but are similarly well-studied. In contrast, the class of nonlinear recurrences has no general unifying theorem. Even the class of quadratic recurrences, those described by a function, F , with total degree 2, has no general characterization. It is for this reason that we turn to computers, and experimental methods in general, to study sequences produced by

nonlinear recurrences. My aim is certainly not to present a unifying theory for nonlinear recurrences, but to study interesting phenomena that arise within this class.

1.1 Thesis Outline

In Chapter 2 we will build up an algorithmic approach for proving global asymptotic stability of rational difference equations (in this field, recurrences are typically referred to as difference equations). The concept of global asymptotic stability is essentially convergence to an equilibrium solution independent of initial conditions; a precise definition will be given below. The basic idea for the algorithmic approach will be to reduce the question of proving global asymptotic stability to that of proving positivity of an associated polynomial. We will begin in Section 2.1 by surveying the definitions and theorems involved in the field of stability of difference equations. Then, in Section 2.2, we will see how to construct a polynomial, P , from a rational difference equation. This polynomial will have the property that if P is positive when its variables are positive then its associated difference equation is globally asymptotically stable. Next, in Section 2.3, I will describe my new algorithm for proving polynomial positivity. A proof-of-concept can be found in 2.4; we apply the algorithm to a specific rational difference equation to prove that it converges to an equilibrium solution regardless of initial conditions. In Section 2.5 we prove global asymptotic stability of many rational difference equations. Finally, in Section 2.6 we briefly summarize the most useful procedures contained in the Maple package associated with Chapter 2.

Chapter 3 contains the study of a three parameter family of nonlinear recurrences, inspired by the famed Somos sequences, that produce integer sequences. We first introduce the Somos recurrences, and summarize various proof techniques used to establish their integrality, in Section 3.1. One of these proof techniques, showing that a sequence generated by a nonlinear recurrence satisfies a linear recurrence, is then used to prove integrality of some sub-cases of the three parameter family in Section 3.2. Lastly, in Section 3.3 we use the Laurent phenomenon to prove integrality of a two parameter sub-family of these nonlinear recurrences. This will prove one direction of a conjecture made in [17].

Finally, in Chapter 4 we consider a more general type of recurrence, that we will call an m -recurrence, that can unexpectedly generate rational numbers. As described above, when generating a sequence $\{a_n\}_{n=1}^{\infty}$ using a recurrence, computing a_n amounts to solving a linear equation in a_n . In this chapter we study what happens when we must solve a degree- m equation, for some integer $m > 1$, in order to compute a_n . In Section 4.1 we consider a degree 2, order 1 example created by manipulating the Somos recurrence. We then consider a generalized version of this example which happens to have connections to elliptic curves and algebraic correspondences. Then, in Section 4.2, we study a higher order, but still degree 2, recurrence obtained from “unfolding” the recurrence in the previous section. We also note a connection with Diophantine equations.

Chapter 2

Global Asymptotic Stability of Rational Difference Equations

In this chapter we will introduce an algorithmic approach to proving *global asymptotic stability (GAS)* of equilibrium points of rational difference equations. This topic has applications to many other fields including biology, economics, and dynamical systems. In application areas, one often studies time-evolving sequences produced by recurrences with the goal of discovering end behavior of the sequence, given some initial conditions. Essentially, when the sequence converges given *any* reasonable initial conditions, we say that it is GAS. In Section 2.1, I will state the precise definition for GAS, as well as introduce all of the other definitions necessary to study stability of difference equations. In order to frame my algorithm within the current theory, Section 2.1 also includes a few relevant theorems that can be used to prove GAS. The final theorem in this section will be used as the basis for my algorithm, which is presented in Sections 2.2 and 2.3. In the algorithm I first reduce the problem of GAS to the problem of proving that a particular polynomial is positive. Then, in Section 2.3, I explain my new algorithm to prove that a multivariate polynomial is positive (when all of its variables are taken to be positive). Next, Section 2.4 contains a proof-of-concept that my algorithm is indeed applicable to prove GAS. Finally, in Section 2.6, I present the most useful commands in the Maple package that I have created to accompany this chapter of my thesis.

2.1 Introduction to Stability of Rational Difference Equations

2.1.1 Definitions

Following the various works of Ladas, et. al. [3, 22, 24], we state a few standard definitions and theorems in the study of difference equations and stability.

Definition 2.1.1. A *difference equation* (of order $k + 1$) is an equation of the form

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}) \quad (2.1)$$

where the function $F(u_0, u_1, \dots, u_k)$ maps I^{k+1} to I , for some interval $I \subseteq \mathbb{R}$.

Typically, we will take I to be $[0, \infty)$ or $(0, \infty)$. In addition, the function F will often be a rational function, and in that case will be denoted R for emphasis.

Given a function F we say that a *solution* of (2.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ which satisfies (2.1). One can also think of a solution, $\{x_n\}_{n=-k}^{\infty}$, as being associated to the specific initial conditions $\{x_{-k}, \dots, x_0\}$ created by repeatedly applying F . If a solution is constant, $x_n = \bar{x}$, for all $n \geq -k$ then we say that the solution is an *equilibrium solution*, and \bar{x} is called an *equilibrium point*, or simply an *equilibrium* of F . In practice, we find the equilibria by solving the equation $\bar{x} = F(\bar{x}, \dots, \bar{x})$, and taking the solutions which lie in the interval I .

The main topic to be investigated in this chapter is end behavior, specifically *stability*, of a solution of a given difference equation. There are various notions of stability that will now be defined.

Definition 2.1.2. An equilibrium point, \bar{x} , of (2.1) is said to be

1. *locally stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution to (2.1) with the property that

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \delta$$

then $|x_n - \bar{x}| < \varepsilon$ for all $n \geq 0$.

2. *locally asymptotically stable (LAS)* if \bar{x} is locally stable, and if there exists a $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution to (2.1) with the property that

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \gamma$$

then

$$\lim_{n \rightarrow \infty} x_n = \bar{x}$$

3. a *global attractor* if for every solution, $\{x_n\}_{n=-k}^{\infty}$, of (2.1) we have

$$\lim_{n \rightarrow \infty} x_n = \bar{x}$$

4. *globally asymptotically stable (GAS)* if \bar{x} is a global attractor, and \bar{x} is locally stable.
5. *unstable* if \bar{x} is not locally stable.

The goal in this chapter of my thesis is to present an algorithm to prove GAS. Since GAS implies LAS, the first step must be to prove LAS (since, if a difference equation is not LAS it can't be GAS). The *linearized stability theorem* provides easily verifiable criteria for local asymptotic stability. In order to state the theorem we must first define the linearized difference equation and characteristic equation. First, suppose that F is continuously differentiable in some open neighborhood of \bar{x} (when $F = R$ is a rational function, this condition is clearly satisfied). Let

$$q_i := \frac{\partial F}{\partial u_i}(\bar{x}, \dots, \bar{x})$$

for $i = 0, 1, \dots, k$, be the partial derivatives of $F(u_0, \dots, u_k)$ w.r.t u_i evaluated at the equilibrium, \bar{x} , of (2.1). Using these q_i we create the *linearized equation of (2.1) about the equilibrium point \bar{x}* ,

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + \cdots + q_k y_{n-k}, \quad n = 0, 1, \dots,$$

and then the *characteristic equation of (2.1) about \bar{x}* ,

$$\lambda^{k+1} - q_0 \lambda^k - q_1 \lambda^{k-1} - \cdots - q_{k-1} \lambda - q_k = 0. \quad (2.2)$$

Now we can state the theorem as found in [3, 7, 15, 24, 28].

Theorem 2.1.1 (Linearized Stability Theorem). *Assume the function F is a continuously differentiable function defined in some neighborhood of an equilibrium point \bar{x} . Then the following statements are true:*

1. *When all the roots of (2.2) have absolute value less than one, then the equilibrium point \bar{x} of (2.1) is locally asymptotically stable.*
2. *If at least one root of (2.2) has absolute value greater than one, then the equilibrium point \bar{x} of (2.1) is unstable.*

For difference equations with order 2, 3, or 4 (i.e., when $k = 1, 2, 3$) there are necessary and sufficient conditions for 1 in Theorem 2.1.1.

Theorem 2.1.2. *Assume $a_1, a_0 \in \mathbb{R}$. Then a necessary and sufficient condition for all the roots of the equation*

$$\lambda^2 + a_1\lambda + a_0 = 0$$

to lie inside the unit disk is

$$|a_1| < 1 + a_0 < 2$$

Theorem 2.1.3. *Assume $a_2, a_1, a_0 \in \mathbb{R}$. Then a necessary and sufficient condition for all the roots of the equation*

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

to lie inside the unit disk is

$$|a_2 + a_0| < 1 + a_1, \quad |a_2 - 3a_0| < 3 - a_1, \quad a_0^2 + a_1 - a_0a_2 < 1$$

Theorem 2.1.4. *Assume $a_3, a_2, a_1, a_0 \in \mathbb{R}$. Then a necessary and sufficient condition for all the roots of the equation*

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

to lie inside the unit disk is

$$|a_1 + a_3| < 1 + a_0 + a_2, \quad |a_1 - a_3| < 2(1 - a_0), \quad a_2 - 3a_0 < 3$$

and

$$a_0 + a_2 + a_0^2 + a_1^2 + a_0^2a_2 + a_0a_3^2 < 1 + 2a_0a_2 + a_1a_3 + a_0a_1a_3 + a_0^3$$

In addition to Theorems 2.1.2 - 2.1.4, the following is a general sufficient condition for 1 in Theorem 2.1.1 (see [22] (p. 12)).

Theorem 2.1.5. *Assume that $q_0, q_1, \dots, q_k \in \mathbb{R}$ are such that*

$$|q_0| + |q_1| + \dots + |q_k| < 1.$$

Then all roots of (2.2) lie inside the unit disk.

In contrast to local asymptotic stability which is easy to verify (except in the case that there are roots of (2.2) with norm 1), global asymptotic stability has no similarly general necessary and sufficient conditions. There are a handful of theorems, providing sufficient conditions, that have been used to verify the global asymptotic stability of many specific difference equations. However, given a difference equation defined by the function F , it is not always obvious which theorem to apply. For the sake of completeness, the next section will discuss a selection of these theorems. The method I will give later in this chapter will only rely on Theorem 2.1.10.

2.1.2 Global Asymptotic Stability Theorems

The following theorem was first stated in [6] as a generalization of a theorem in [5].

Theorem 2.1.6. *Let I be an interval of real numbers and let $F \in C(I^{k+1}, I)$ (the set of continuous functions from I^{k+1} to I). Assume the following three conditions are satisfied:*

1. *F is increasing in each of its arguments,*
2. *$F(z_1, \dots, z_{k+1})$ is strictly increasing in each of the arguments $z_{i_1}, z_{i_2}, \dots, z_{i_l}$ where $1 \leq i_1 \leq i_2 \leq \dots \leq i_l \leq k+1$, and the arguments i_1, i_2, \dots, i_l are relatively prime,*
3. *Every point c in I is an equilibrium point of (2.1).*

Then every solution of (2.1) has a finite limit.

Notice here that there is no mention of what this limit is. The next result, due to Hatus and Bolis in [16], can also be found in [22] as Theorem 2.6.2. It states sufficient conditions for \bar{x} to be a global attractor.

Theorem 2.1.7. *Let I be an open interval of real numbers, let $F \in C(I^{k+1}, I)$, and let $\bar{x} \in I$ be an equilibrium point of (2.1). Assume that F satisfies the following conditions:*

1. F is increasing in each of its arguments
2. F satisfies the negative feedback property

$$(u - \bar{x})(F(u, u, \dots, u) - u) < 0, \quad \text{for all } u \in I \setminus \{\bar{x}\}$$

Then the equilibrium point \bar{x} is a global attractor for (2.1).

In the following theorem, found in [22], we see sufficient conditions for \bar{x} to be a global attractor of an order 2 difference equation of the form $x_{n+1} = x_n f(x_n, x_{n-1})$.

Theorem 2.1.8. *Assume that the following conditions hold:*

1. $f \in C((0, \infty) \times (0, \infty), (0, \infty))$
2. $f(x, y)$ is decreasing in x and increasing in y
3. $xf(x, x)$ is strictly increasing in x
4. The equation

$$x_{n+1} = x_n f(x_n, x_{n-1}), \quad n = 0, 1, \dots \tag{2.3}$$

has a unique positive equilibrium \bar{x}

Then \bar{x} is a global attractor of all positive solutions of (2.3).

The next theorem is from [24], a book by Kulenović and Ladas which summarizes all known results for second order rational difference equations (with linear numerator and denominator). This theorem introduces the notation $M_i(m, M)$ and $m_i(m, M)$, which is used in many similar theorems.

Theorem 2.1.9. *Let $[a, b]$ be a closed and bounded interval of real numbers, and let $F \in C([a, b]^{k+1}, [a, b])$ satisfy the following conditions*

1. *The function $F(z_1, \dots, z_{k+1})$ is monotonic in each of its arguments.*

2. For each $m, M \in [a, b]$ and for each $i \in \{1, \dots, k+1\}$ define

$$M_i(m, M) = \begin{cases} M, & \text{if } F \text{ is increasing in } z_i \\ m, & \text{if } F \text{ is decreasing in } z_i \end{cases} \quad (2.4)$$

and

$$m_i(m, M) = M_i(M, m) \quad (2.5)$$

and assume that if (m, M) is a solution of the system

$$M = F(M_1(m, M), \dots, M_{k+1}(m, M))$$

$$m = F(m_1(m, M), \dots, m_{k+1}(m, M))$$

then $M = m$.

Then there exists exactly one equilibrium, \bar{x} , of the equation

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots$$

and every solution of this equation converges to \bar{x} .

The final theorem in this section, which will be utilized in the GAS algorithm in Section 2.3, is first presented in a paper by Kruse and Nesemann [23]. It will be stated in a slightly different manner than it appears in their paper, using the notation we have established in this thesis. First, it will be necessary to consider the difference equation associated to a function F in vector form. Let $G : \mathbb{R}^{k+1} \rightarrow \mathbb{R}^{k+1}$ be defined from F as

$$G(\mathcal{X}_n) = G \left(\begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-k} \end{bmatrix} \right) = \begin{bmatrix} F(x_n, \dots, x_{n-k}) \\ x_n \\ \vdots \\ x_{n-k+1} \end{bmatrix} = \mathcal{X}_{n+1}. \quad (2.6)$$

In the case that $F = R$ is rational we denote this vector mapping by Q instead of G . Note that this transformation from F to G essentially creates an order 1 mapping out of an order $k+1$ mapping. In addition, G is now a map that can be composed with itself, so

$$\mathcal{X}_n = G^n(\mathcal{X}_0)$$

where $\mathcal{X}_0 = \langle x_0, \dots, x_{-k} \rangle$ is the vector of initial conditions. Now we can state the theorem.

Theorem 2.1.10 (Kruse, Neseemann 1999). *Let $\|\cdot\|$ denote the Euclidean norm (i.e., $\|(a, b)\| = \sqrt{a^2 + b^2}$). Let \mathbb{S} denote either $[0, \infty)$ or $(0, \infty)$ (the function G will necessitate which). Let $G : \mathbb{S}^{k+1} \rightarrow \mathbb{S}^{k+1}$ be a continuous mapping of the form (2.6) with a unique fixed point $\bar{\mathcal{X}} \in \mathbb{S}^{k+1}$. Suppose for the discrete dynamic system*

$$\mathcal{X}_{n+1} = G(\mathcal{X}_n), \quad n = 0, 1, 2, \dots \quad (2.7)$$

there exists an integer $K \geq 1$ such that the K^{th} iterate of G satisfies

$$\|G^K(\mathcal{X}) - \bar{\mathcal{X}}\| < \|\mathcal{X} - \bar{\mathcal{X}}\| \quad \text{for all } \mathcal{X} \in \mathbb{S}^{k+1}, \mathcal{X} \neq \bar{\mathcal{X}}. \quad (2.8)$$

Then $\bar{\mathcal{X}}$ is GAS with respect to the norm $\|\cdot\|$.

First, let me point out that this integer K tells us which power of G is a contraction with respect to $\bar{\mathcal{X}}$, i.e., G^K shrinks distances to $\bar{\mathcal{X}}$. This gives an intuitive reason for $\|G^K(\mathcal{X}) - \bar{\mathcal{X}}\| < \|\mathcal{X} - \bar{\mathcal{X}}\|$ to imply global asymptotic stability.

Before giving the proof let me first point out that the various definitions of stability, as they are stated in Definition 2.1.2, do not quite apply here because our unique fixed point (or equilibrium) is a vector rather than a scalar. However, Definition 2.1.2, can be translated to the vector case. The recurrence is (2.7), the equilibrium is a vector solution to the equation $G(\bar{\mathcal{X}}) = \bar{\mathcal{X}}$, and the order of the recurrence, $k + 1$, is 1 (so $k = 0$). Other than these minor changes, a word for word translation of Definition 2.1.2 is what we mean by $\bar{\mathcal{X}}$ being GAS in Theorem 2.1.10. With this in mind we now restate the proof from [23] following our notation and including a few more details.

Proof. To prove that $\bar{\mathcal{X}}$ is GAS we must show two things: first $\bar{\mathcal{X}}$ is locally stable, and second $\bar{\mathcal{X}}$ is a global attractor. To prove that $\bar{\mathcal{X}}$ is locally stable we must show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\|\mathcal{X}_0 - \bar{\mathcal{X}}\| < \delta$ then

$$\|\mathcal{X}_n - \bar{\mathcal{X}}\| = \|G^n(\mathcal{X}_0) - \bar{\mathcal{X}}\| < \varepsilon \quad \text{for all } n \geq 0.$$

To find δ , given ε , we will use the fact that G is continuous, and thus G^i , the composition of G with itself i times, is continuous for all i (in particular for $0 \leq i \leq K - 1$). Thus,

for every $\varepsilon > 0$ there exist $\delta_0, \delta_1, \dots, \delta_{K-1} > 0$ such that

$$\|\mathcal{X}_0 - \bar{\mathcal{X}}\| < \delta_i \implies \|G^i(\mathcal{X}_0) - \bar{\mathcal{X}}\| < \varepsilon.$$

Notice that $\delta_0 = \varepsilon$ since

$$\|\mathcal{X}_0 - \bar{\mathcal{X}}\| < \delta_0 = \varepsilon \implies \|G^0(\mathcal{X}_0) - \bar{\mathcal{X}}\| = \|\mathcal{X}_0 - \bar{\mathcal{X}}\| < \varepsilon.$$

Now let $\delta := \min_{0 \leq i \leq K-1} \{\delta_i\} \leq \varepsilon$ and assume $\|\mathcal{X}_0 - \bar{\mathcal{X}}\| < \delta$. Then from continuity we have

$$\|G^i(\mathcal{X}_0) - \bar{\mathcal{X}}\| < \varepsilon \text{ for } 0 \leq i \leq K-1. \quad (2.9)$$

Also, we have assumed in (2.8) that

$$\|G^K(\mathcal{X}_0) - \bar{\mathcal{X}}\| < \|\mathcal{X}_0 - \bar{\mathcal{X}}\| < \delta \leq \varepsilon. \quad (2.10)$$

So now we can proceed by induction. We have just shown the base case,

$$\|G^i(\mathcal{X}_0) - \bar{\mathcal{X}}\| < \varepsilon \text{ for } 0 \leq i \leq K-1 \text{ and } \|G^K(\mathcal{X}_0) - \bar{\mathcal{X}}\| < \delta (\leq \varepsilon).$$

For some $N \geq 1$, assume as the induction hypothesis that,

$$\|G^{NK+i}(\mathcal{X}_0) - \bar{\mathcal{X}}\| < \varepsilon \text{ for } 0 \leq i \leq K-1 \text{ and } \|G^{(N+1)K}(\mathcal{X}_0) - \bar{\mathcal{X}}\| < \delta (\leq \varepsilon).$$

To complete the inductive proof we need to show

$$\|G^{(N+1)K+i}(\mathcal{X}_0) - \bar{\mathcal{X}}\| < \varepsilon \text{ for } 0 \leq i \leq K-1 \text{ and } \|G^{(N+2)K}(\mathcal{X}_0) - \bar{\mathcal{X}}\| < \delta (\leq \varepsilon).$$

By continuity, and because we know $\|G^{(N+1)K}(\mathcal{X}_0) - \bar{\mathcal{X}}\| < \delta$ from the induction hypothesis, we can apply G^i to $G^{(N+1)K}(\mathcal{X}_0)$ to get:

$$\|G^i(G^{(N+1)K}(\mathcal{X}_0)) - \bar{\mathcal{X}}\| = \|G^{(N+1)K+i}(\mathcal{X}_0) - \bar{\mathcal{X}}\| < \varepsilon \text{ for } 0 \leq i \leq K-1.$$

As in the base case, we use the assumption (2.8) to see

$$\|G^K(G^{(N+1)K}(\mathcal{X}_0)) - \bar{\mathcal{X}}\| = \|G^{(N+2)K}(\mathcal{X}_0) - \bar{\mathcal{X}}\| < \|G^{(N+1)K}(\mathcal{X}_0) - \bar{\mathcal{X}}\|,$$

which again is less than δ (and so, less than ε) by the induction hypothesis. This completes the proof of local stability. We still need to show that $\bar{\mathcal{X}}$ is a global attractor.

Let $\mathcal{X}_0 \neq \bar{\mathcal{X}}$ be fixed initial conditions for the discrete dynamic system (2.7). For fixed $0 \leq j \leq K - 1$, let $\mathcal{Y}_N^j := \mathcal{X}_{NK+j}$ and consider the subsequence $\{\mathcal{Y}_N^j\}_{N=0}^\infty$ of $\{\mathcal{X}_n\}_{n=-1}^\infty$. For the remainder of the proof assume that $j \in \{0, \dots, K - 1\}$ is arbitrary, but fixed. Also notice that

$$G^K(\mathcal{Y}_N^j) = \mathcal{Y}_{N+1}^j,$$

and therefore by (2.8)

$$\|\mathcal{Y}_{N+1}^j - \bar{\mathcal{X}}\| < \|\mathcal{Y}_N^j - \bar{\mathcal{X}}\|. \quad (2.11)$$

Then, since $\|\mathcal{Y}_N^j - \bar{\mathcal{X}}\|$ is strictly decreasing in N , and bounded from below by 0, we know that this sequence of norms must have a limit:

$$\lim_{N \rightarrow \infty} \|\mathcal{Y}_N^j - \bar{\mathcal{X}}\| =: L \geq 0.$$

Also from (2.11) we know that the sequence $\{\mathcal{Y}_N^j\}_{N=0}^\infty$ is bounded, in fact it is easily seen that $\|\mathcal{Y}_N^j\| < \|\mathcal{Y}_0^j\| + 2\|\bar{\mathcal{X}}\|$. Therefore it has a convergent subsequence, $\{\mathcal{Y}_{m(N)}^j\}_{N=0}^\infty$. Let $A := \lim_{N \rightarrow \infty} \mathcal{Y}_{m(N)}^j$. Then

$$L = \lim_{N \rightarrow \infty} \|\mathcal{Y}_N^j - \bar{\mathcal{X}}\| = \lim_{N \rightarrow \infty} \|\mathcal{Y}_{m(N)}^j - \bar{\mathcal{X}}\| = \|A - \bar{\mathcal{X}}\|.$$

Finally, because G is assumed to be continuous we see that

$$L = \lim_{N \rightarrow \infty} \|\mathcal{Y}_{m(N)+1}^j - \bar{\mathcal{X}}\| = \lim_{N \rightarrow \infty} \|G^K(\mathcal{Y}_{m(N)}^j) - \bar{\mathcal{X}}\| = \|G^K(A) - \bar{\mathcal{X}}\|.$$

Then we must have $A = \bar{\mathcal{X}}$, otherwise there is a contradiction to (2.8). Therefore, $L = 0$ and so $\lim_{N \rightarrow \infty} \mathcal{Y}_N^j = \bar{\mathcal{X}}$. We chose $j \in \{0, \dots, K - 1\}$ to be arbitrary, so $\lim_{N \rightarrow \infty} \mathcal{X}_n = \bar{\mathcal{X}}$. \square

Next we will see how this theorem can be used to create a global asymptotic stability proof algorithm.

2.2 From Global Asymptotic Stability to Polynomial Positivity

In this section we will see how to reduce the question of global asymptotic stability of a rational difference equation to a question about an associated polynomial being positive.

Throughout this section assume that we have fixed a rational difference equation,

$$x_{n+1} = R(x_n, \dots, x_{n-k}), \quad (2.12)$$

of order $k + 1$, with a unique equilibrium \bar{x} . Also assume that R is a rational function with positive coefficients, so $R : [0, \infty)^{k+1} \rightarrow [0, \infty)$, and \bar{x} is non-negative (if there is no constant term in the denominator of R we cannot allow 0 to be in the domain, so $R : (0, \infty)^{k+1} \rightarrow (0, \infty)$, and \bar{x} must be strictly positive). In order to apply Theorem 2.1.10 we must think of (2.12) and its equilibrium in their vector forms. For example, if $x_{n+1} = R(x_n, x_{n-1}) = \frac{4+x_n}{1+x_{n-1}}$ then

$$Q \left(\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} \right) = \begin{bmatrix} \frac{4+x_n}{1+x_{n-1}} \\ x_n \end{bmatrix},$$

and $\bar{\mathcal{X}} = \langle 2, 2 \rangle$. In this case $k = 1$, so $R : [0, \infty)^2 \rightarrow [0, \infty)$, and $Q : [0, \infty)^2 \rightarrow [0, \infty)^2$.

The goal will be to find a positive integer, K , which satisfies (2.8). Motivated by this goal, we will construct the following polynomial, given specific Q , $\bar{\mathcal{X}}$, and K (assume we have conjectured some value for K):

$$P_{Q, \bar{\mathcal{X}}, K}(\mathcal{X}) = \text{numerator} \left(\|\mathcal{X} - \bar{\mathcal{X}}\|^2 - \|Q^K(\mathcal{X}) - \bar{\mathcal{X}}\|^2 \right). \quad (2.13)$$

Consider the implication of $P_{Q, \bar{\mathcal{X}}, K} > 0$ for $\mathcal{X} \geq 0$ (or > 0 , both componentwise), and $\mathcal{X} \neq \bar{\mathcal{X}}$.

$$\begin{aligned} 0 &< \text{numerator} \left(\|\mathcal{X} - \bar{\mathcal{X}}\|^2 - \|Q^K(\mathcal{X}) - \bar{\mathcal{X}}\|^2 \right) \\ \implies 0 &< \|\mathcal{X} - \bar{\mathcal{X}}\|^2 - \|Q^K(\mathcal{X}) - \bar{\mathcal{X}}\|^2 \\ \implies \|Q^K(\mathcal{X}) - \bar{\mathcal{X}}\|^2 &< \|\mathcal{X} - \bar{\mathcal{X}}\|^2 \\ \implies \|Q^K(\mathcal{X}) - \bar{\mathcal{X}}\| &< \|\mathcal{X} - \bar{\mathcal{X}}\|. \end{aligned} \quad (2.14)$$

Of course, the first implication, undoing the numerator from line 1 to line 2, in general will not preserve an inequality since the denominator may be negative. However, because we are squaring the Euclidean norm, the common denominator is always a product of sums of squares. Taking the numerator is then equivalent to multiplying both sides by the denominator, a positive quantity, which will not change the direction

of the inequality. Notice that the final implicant, (2.14), is simply (2.8), so proving $P_{Q,\bar{x},K} > 0$ for some K implies that \bar{x} is GAS for the rational difference equation R . An algorithm for proving positivity will be shown in Section 2.3. Also note that whenever the function Q and equilibrium are clear from context, they will be omitted from the subscript of P .

Consider the above example, $x_{n+1} = \frac{4+x_n}{1+x_{n-1}}$, with the equilibrium $\bar{x} = 2$. If we let $K = 1$ then the polynomial is:

$$\begin{aligned}
P_1(\langle x_1, x_2 \rangle) &= \text{numerator} \left(\|\mathcal{X} - \bar{\mathcal{X}}\|^2 - \|Q^K(\mathcal{X}) - \bar{\mathcal{X}}\|^2 \right) \\
&= \text{numerator} \left(\|\langle x_1, x_2 \rangle - \langle 2, 2 \rangle\|^2 - \left\| \left\langle \frac{4+x_1}{1+x_2}, x_1 \right\rangle - \langle 2, 2 \rangle \right\|^2 \right) \\
&= \text{numerator} \left((x_1 - 2)^2 + (x_2 - 2)^2 - \left(\left(\frac{4+x_1}{1+x_2} - 2 \right)^2 + (x_1 - 2)^2 \right) \right) \\
&= \text{numerator} \left(-\frac{(7x_2^2 + 2x_2^3 - x_2^4 - 12x_2 + 4x_1 + x_1^2 - 4x_1x_2)}{(1+x_2)^2} \right) \\
&= -x_1^2 + 4x_1x_2 - 4x_1 + x_2^4 - 2x_2^3 - 7x_2^2 + 12x_2.
\end{aligned}$$

This polynomial is not positive since the coefficient on the highest power of x_1 is negative. However, this example was only meant to show the process to get a polynomial from Q , \bar{x} , and K . For the correct K value and proof see Section 2.4.

For a given R and \bar{x} we know that showing positivity of an associated polynomial implies GAS of \bar{x} for R . We also know, given K , what that polynomial associated to Q and $\bar{\mathcal{X}}$ is. However, we still need to see how to conjecture a reasonable value for K , and then how to prove that the polynomial is indeed positive. We will see how to prove positivity in the next section. Now let's see how to conjecture a reasonable K value given R and \bar{x} using a brute force method. Start with $K = 1$ and apply the following algorithm:

1. Create the polynomial $P_{Q,\bar{x},K}(\mathcal{X})$
2. Apply a minimization technique to the polynomial $P_{Q,\bar{x},K}(\mathcal{X})$ (e.g., simulated annealing, gradient descent, Metropolis-Hastings algorithm, etc.) many times to find approximate local minima of $P_{Q,\bar{x},K}$.

3. (a) If all minima are positive then conjecture that this K works.
- (b) If there is a negative minima then increment K by 1 and go back to step 1.

For ease of computation, and since this is only to conjecture a K , we apply the minimization technique in step 2 to a discrete set of points. We will restrict to a fine mesh with large upper bound. For example, the cartesian product $\times_{i=1}^{k+1} \{\varepsilon, 2\varepsilon, \dots, N\varepsilon\}$, for some large value of N and small value of ε . Then every point in the mesh is a vector of the form $\langle i_1\varepsilon, i_2\varepsilon, \dots, i_{k+1}\varepsilon \rangle$, where $1 \leq i_j \leq N$.

Note that this is not the only possible algorithm for conjecturing a value for K . However, the main result in this chapter is a positivity algorithm, so we will not consider other possible algorithms. One could, in theory, replace step 2 with the following, “Apply the polynomial positivity algorithm found in Section 2.3”. Then step 3 would become, “If the algorithm in step 2 fails, increase K by 1 and go back to step 1, otherwise return K ”. Using this positivity algorithm, once a K value is found, it is also proved to be correct. However, using positivity in step 2 is sometimes not feasible since it often takes more computer memory than the conjecturing algorithm.

2.3 An Algorithm to Prove Positivity of a Multivariate Polynomial

So far, my algorithm to prove global asymptotic stability of a particular rational difference equation has reduced the problem to proving that an associated polynomial is positive. Now the question becomes, how does one prove positivity? The algorithm I will present was inspired by the following definition and theorem found in [21].

Definition 2.3.1. The polynomial $P \in \mathbb{R}[x_1, \dots, x_n]$ is

- *positive* (resp. *non-negative*) from μ iff $\forall x_1 \geq \mu, \dots, x_n \geq \mu, P(x_1, \dots, x_n) > 0$ (resp. $P(x_1, \dots, x_n) \geq 0$).
- *absolutely positive* (resp. *absolutely non-negative*) from μ iff P is positive (resp. non-negative) from μ , and every partial derivative (of any order), P^* , of P is non-negative from μ , i.e., $\forall x_1 \geq \mu, x_2 \geq \mu, \dots, x_n \geq \mu, P^*(x_1, \dots, x_n) \geq 0$.

In addition, we will denote by $\sigma_{\mu_1, \dots, \mu_n}(P)$ the polynomial obtained from P by translating in dimension i by μ_i in the negative direction. In other words, replace x_i by $x_i + \mu_i$ in P for all i . If $\mu_i = \mu$ for all i then we simply write $\sigma_\mu(P)$. Now, I will state a theorem from [21] that gives a necessary and sufficient condition for absolute positivity (and absolute non-negativity).

Theorem 2.3.1 (Hong, Jakuš 1998). *Let P be a non-zero polynomial. Then P is absolutely positive (resp. absolutely non-negative) from μ iff every coefficient in $\sigma_\mu(P)$ is positive, and the constant term is nonzero (resp. non-negative). In particular, if $\mu = 0$ then every coefficient in P is positive and the constant term is nonzero (resp. non-negative).*

Now, it is certainly too much to hope for the polynomials $P_{Q, \bar{x}, K}$ to be *absolutely* positive from zero. Of course, to satisfy Theorem 2.1.10, it is only necessary that they be positive from zero (and possibly zero at a few points). My algorithm will subdivide the positive *orthant* (the region in which all the variables are non-negative), denoted by \mathbb{R}_+^n where n is the number of variables in P , into regions in which P is essentially absolutely positive in some direction (i.e., there is a direction such that the directional derivative is positive).

Example 2.3.1. Consider the paraboloid given by $P = (x - 1)^2 + (y - 2)^2$. The directional derivative, at the point $\langle x, y \rangle$ in the direction $\vec{u} = \langle u_1, u_2 \rangle$, is given by

$$\nabla P(x, y) \cdot \vec{u} = 2(x - 1)u_1 + 2(y - 2)u_2.$$

If we can find a decomposition of \mathbb{R}_+^2 into subsets so that $P(x, y) > 0$ on the boundary of each set, and the directional derivative away from the boundary is positive, this guarantees positivity of $P(x, y)$ on \mathbb{R}_+^2 . I propose the decomposition

$$\begin{aligned} \mathbb{R}_+^2 = & \{(x, y) : 1 \leq x \text{ and } 2 \leq y\} \cup \{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2\} \cup \\ & \cup \{(x, y) : 0 \leq x \leq 1 \text{ and } 2 \leq y\} \cup \{(x, y) : 1 \leq x \text{ and } 0 \leq y \leq 2\}. \end{aligned}$$

Clearly P is positive on the boundary of each of these regions (set $x = 1$ or $y = 2$). Now, if $1 \leq x$ and $2 \leq y$ the direction $\vec{u} = \langle 1, 1 \rangle$ guarantees positivity of the directional

derivative since

$$2(x-1)u_1 + 2(y-2)u_2 = 2x + 2y - 6 > 2 + 4 - 6 = 0.$$

Next, if $0 \leq x \leq 1$ and $0 \leq y \leq 2$ we consider the direction $\vec{u} = \langle -1, -1 \rangle$ and see that

$$2(x-1)u_1 + 2(y-2)u_2 = -2x - 2y + 6 > -2 - 4 + 6 = 0.$$

Finally, if $0 \leq x \leq 1$ and $2 \leq y$ the directional derivative is positive in the direction $\vec{u} = \langle -1, 1 \rangle$:

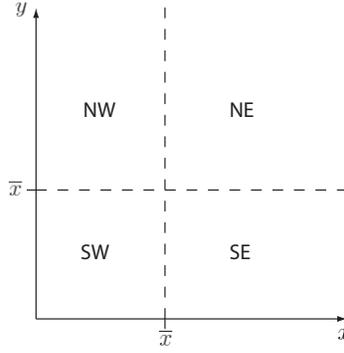
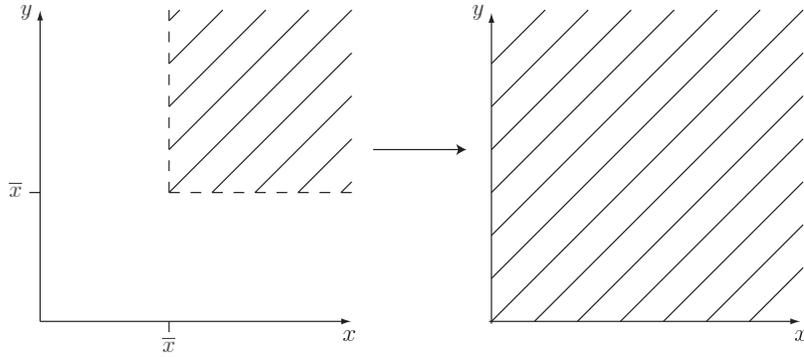
$$2(x-1)u_1 + 2(y-2)u_2 = -2x + 2y - 2 > -2 + 4 - 2 = 0.$$

Similarly, if $1 \leq x$ and $0 \leq y \leq 2$ then the directional vector must be $\vec{u} = \langle 1, -1 \rangle$. \diamond

Since the polynomials we construct while pursuing global asymptotic stability are typically much more complicated than the one found in the previous example, we cannot easily show that a directional derivative is positive. Instead, for each region $S \subset \mathbb{R}_+^n$ we will create a polynomial, $P_S(y)$. This polynomial will have the property that if $P_S(y) \geq 0$ for all $y \in \mathbb{R}_+^n$ then $P(x) \geq 0$ for all $x \in S$. The algorithm will first be described in two dimensions, and later generalized to the n -dimensional case.

Let $P := P(x, y)$ be a polynomial in two variables ($n = 2$). In order to show that $P(x, y) \geq 0$ for $(x, y) \in \mathbb{R}_+^2$, we first cut the positive quadrant into 4 regions as shown in Figure 2.1, where \bar{x} is some positive number. In the case that $P = P_{Q, \bar{x}, K}$ as in Section 2.2, \bar{x} will be the equilibrium point of the rational difference equation used to create P .

For each of these four regions we create a new polynomial from P by transforming the region into \mathbb{R}_+^2 , and making the corresponding variable substitutions. See Figures 2.2 - 2.4 for the region transformations (transforming SE is analogous to transforming NW by permuting the x and y axes). Based on these region transformations we see

Figure 2.1: Cutting \mathbb{R}_+^2 into 4 regionsFigure 2.2: Transforming NE to \mathbb{R}_+^2

that the associated polynomials are given by

$$\begin{aligned}
 P_{NE}(x, y) &= \sigma_{\bar{x}}(P) = P(x + \bar{x}, y + \bar{x}), \\
 P_{SW}(x, y) &= \sigma_{\frac{1}{\bar{x}}}\left(P\left(\frac{1}{x}, \frac{1}{y}\right) x^{d_x} y^{d_y}\right) \\
 &= P\left(\frac{1}{x + \frac{1}{\bar{x}}}, \frac{1}{y + \frac{1}{\bar{x}}}\right) \left(x + \frac{1}{\bar{x}}\right)^{d_x} \left(y + \frac{1}{\bar{x}}\right)^{d_y}, \\
 P_{NW}(x, y) &= \sigma_{\frac{1}{\bar{x}}, \bar{x}}\left(P\left(\frac{1}{x}, y\right) x^{d_x}\right) = P\left(\frac{1}{x + \frac{1}{\bar{x}}}, y + \bar{x}\right) \left(x + \frac{1}{\bar{x}}\right)^{d_x}, \\
 P_{SE}(x, y) &= \sigma_{\bar{x}, \frac{1}{\bar{x}}}\left(P\left(x, \frac{1}{y}\right) y^{d_y}\right) = P\left(x + \bar{x}, \frac{1}{y + \frac{1}{\bar{x}}}\right) \left(y + \frac{1}{\bar{x}}\right)^{d_y}.
 \end{aligned} \tag{2.15}$$

The change of variables in P before applying σ , inverting the variables or not, is self explanatory based on the transformation of the associated region. However, we must also multiply by x^{d_x} and/or y^{d_y} (where $d_z =$ the degree of z in P for $z = x, y$) as needed before applying the σ shift operator so that the resulting P_{\square} is still a polynomial. When

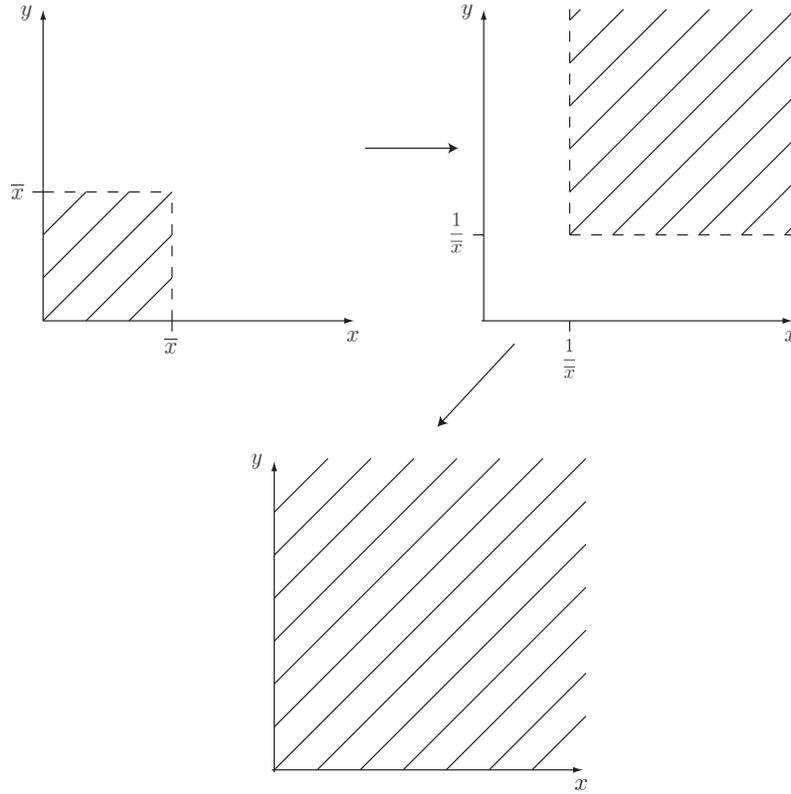


Figure 2.3: Transforming SW to \mathbb{R}_+^2

talking generally about one of these polynomials, we will denote it by P_\square , where the \square can refer to an arbitrary region. Note that if $\bar{x} = 0$ we will only consider the *NE* region, thus avoiding translating by $\frac{1}{\bar{x}} = \frac{1}{0}$.

Before we continue the algorithm by giving criteria to test positivity of the polynomials in (2.15) we must see why proving positivity of all P_\square will be enough to prove positivity of $P(x, y)$ for all $(x, y) \in \mathbb{R}_+^2$.

Proposition 2.3.2. *Let $P(x, y)$ be a polynomial, $d_x = \deg_x(P)$, $d_y = \deg_y(P)$, and $\bar{x} > 0$. Consider the polynomials $P_{NE}, P_{SW}, P_{NW}, P_{SE}$ as defined in (2.15). If these four polynomials are all non-negative from 0 then $P(x, y)$ is non-negative from 0.*

Proof. For each of the four polynomials we will see that positivity for $(x, y) \in \mathbb{R}_+^2$ implies positivity of $P(x, y)$ in the corresponding region.

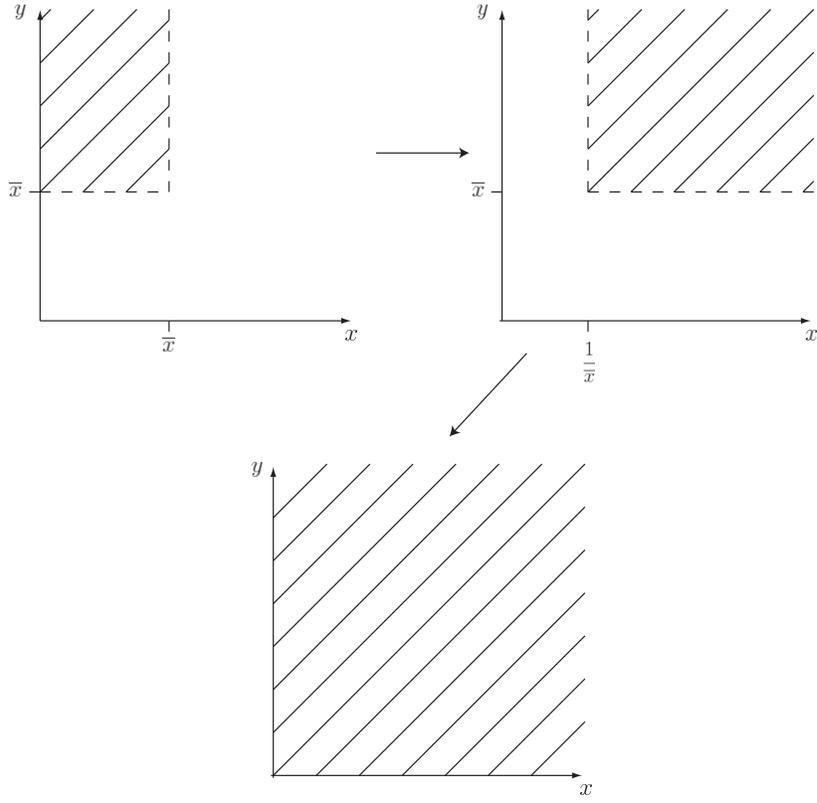


Figure 2.4: Transforming NW to \mathbb{R}_+^2

If $P_{NE}(x, y) \geq 0$ for $(x, y) \in \mathbb{R}_+^2$: Then by definition of $P_{NE}(x, y)$ we have

$$P_{NE}(x, y) = P(x + \bar{x}, y + \bar{x}) \geq 0 \quad \text{for } x \geq 0, y \geq 0.$$

Let $x' := x + \bar{x}$ and $y' := y + \bar{x}$, then

$$P(x', y') \geq 0 \quad \text{for } x' = x + \bar{x} \geq \bar{x}, \text{ and } y' = y + \bar{x} \geq \bar{x}.$$

This says precisely that $P(x, y) \geq 0$ in the region NE .

If $P_{SW}(x, y) \geq 0$ for $(x, y) \in \mathbb{R}_+^2$: Again, by definition of $P_{SW}(x, y)$

$$P_{SW}(x, y) = P\left(\frac{1}{x + \frac{1}{\bar{x}}}, \frac{1}{y + \frac{1}{\bar{x}}}\right) \left(x + \frac{1}{\bar{x}}\right)^{d_x} \left(y + \frac{1}{\bar{x}}\right)^{d_y} \geq 0$$

$$\text{for } x \geq 0, y \geq 0.$$

Following the previous case we first substitute $x' := x + \frac{1}{\bar{x}}$ and $y' := y + \frac{1}{\bar{x}}$ to get

$$P\left(\frac{1}{x'}, \frac{1}{y'}\right) (x')^{d_x} (y')^{d_y} \geq 0 \quad \text{for } x' = x + \frac{1}{\bar{x}} \geq \frac{1}{\bar{x}}, \text{ and } y' = y + \frac{1}{\bar{x}} \geq \frac{1}{\bar{x}}.$$

Since we are only interested in the region for which x' and y' are both strictly positive we may cancel the $(x')^{d_x} (y')^{d_y}$ without reversing the inequality. We also make a second substitution letting $x'' := \frac{1}{x'}$ and $y'' := \frac{1}{y'}$. Now we see that

$$P(x'', y'') \geq 0 \quad \text{for } 0 < x'' = \frac{1}{x'} \leq \bar{x}, \text{ and } 0 < y'' = \frac{1}{y'} \leq \bar{y}$$

which is simply $P(x, y) \geq 0$ in the region SW .

If $P_{NW}(x, y) \geq 0$ for $(x, y) \in \mathbb{R}_+^2$: From the definition in (2.15) this means

$$P_{NW}(x, y) = P\left(\frac{1}{x + \frac{1}{\bar{x}}}, y + \bar{x}\right) \left(x + \frac{1}{\bar{x}}\right)^{d_x} \geq 0 \quad \text{for } x \geq 0, y \geq 0.$$

As in the SW case, we will make two substitutions. The first being $x' := x + \frac{1}{\bar{x}}$ and $y' := y + \bar{x}$. This gives us

$$P\left(\frac{1}{x'}, y'\right) (x')^{d_x} \geq 0 \quad \text{for } x' = x + \frac{1}{\bar{x}} \geq \frac{1}{\bar{x}}, \text{ and } y' = y + \bar{x} \geq \bar{x}.$$

Again, we may cancel the $(x')^{d_x}$ without reversing the inequality since x' is strictly positive in the region in question. Finally, we make our second substitution, $x'' := \frac{1}{x'}$ (there is no second substitution for y') which yields

$$P(x'', y') \geq 0 \quad \text{for } 0 < x'' = \frac{1}{x'} \leq \bar{x}, \text{ and } y' \geq \bar{x}.$$

Therefore, $P(x, y) \geq 0$ in the region NW .

If $P_{SE}(x, y) \geq 0$ for $(x, y) \in \mathbb{R}_+^2$: This case is analogous to the NW case by interchanging the roles of x and y .

In each of the four cases positivity of the polynomial corresponds to positivity of $P(x, y)$ in the corresponding region. \square

To prove positivity of each of the P_{\square} we will test two criteria, neither using anything more powerful than high school algebra.

PosCoeffs: From Theorem 2.3.1, if all coefficients, including the constant term, of P_{\square} are non-negative then $P_{\square}(x, y) \geq 0$ for $(x, y) \in \mathbb{R}_+^2$.

SubPoly: If the only negative coefficient in P_{\square} (including the constant term) is on the xy term then we check whether the binary quadratic form,

$$ax^2 + bxy + cy^2 \tag{2.16}$$

where a , b , and c are coefficients of their respective terms in P_{\square} , is positive definite (i.e., is positive for all $(x, y) \neq 0$) using its discriminant. The binary quadratic form discriminant of (2.16) is defined to be $d = 4ac - b^2$ [30]. If $a, d > 0$, then (2.16) is positive. Then, if this “sub-polynomial” of P_{\square} is positive, P_{\square} itself is positive (since the other coefficients are positive). Notice that this may not be the discriminant most are familiar with. A further discussion of why this is taken to be the discriminant can be found when the n -dimensional positivity algorithm is summarized later in this section.

We also have an easy way to test whether $P(x, y) < 0$ for some $(x, y) \in \mathbb{R}_+^2$ by checking the leading coefficient (the coefficient on the highest degree term) and constant term.

LCoeff: The leading coefficient must be positive, otherwise the polynomial eventually tends to negative infinity in some direction.

Const: Similarly, the constant term must be positive, otherwise the polynomial is negative in a neighborhood of the origin.

For each region \square , if P_{\square} passes one of **PosCoeffs** or **SubPoly** then, by Proposition 2.3.2, $P(x, y) \geq 0$ in the region \square . If P_{\square} fails one of **LCoeff** or **Const** then we output **false** immediately because we know that there are points in region \square for which $P(x, y)$ is negative. However, for some region \square , if P_{\square} has too many negative coefficients, its leading coefficient is positive, and its constant term is positive, then we must do more tests to establish positivity of P_{\square} on \mathbb{R}_+^2 .

We would like to subdivide our original region (NE, NW, SE, or SW) into finitely many pieces and try again. However, there isn’t an obvious way to do this since, except for SW, the regions are infinite, and we have used our “obvious” cutpoint, \bar{x} . So instead, we will first map the infinite region into a finite rectangle with lower left corner at the

origin (see figures 2.5 and 2.6) and create a new polynomial $P'_\square(x, y)$ from $P(x, y)$ for each now finite region. We will then subdivide this finite region in order to prove that $P'_\square(x, y) \geq 0$ on the region in which it is defined. These new polynomials will be defined in the following manner based on their corresponding region transformations.

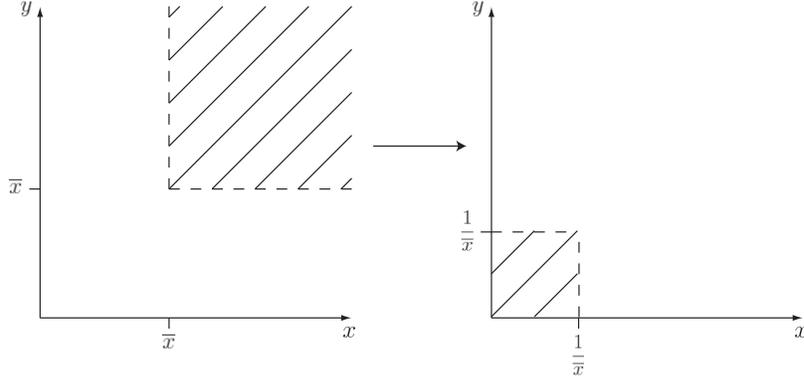


Figure 2.5: Transforming NE to finite rectangle

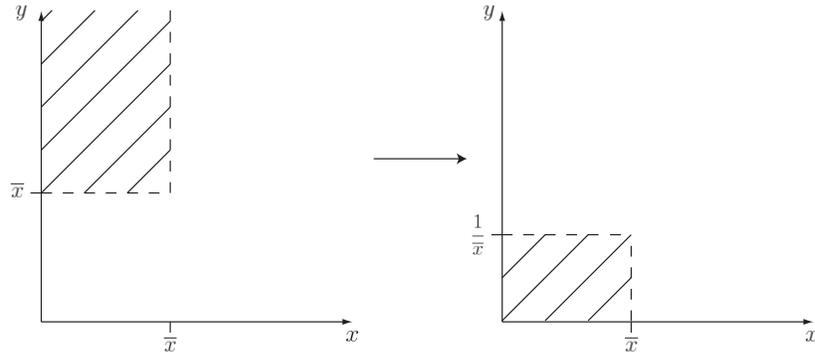


Figure 2.6: Transforming NW to finite rectangle (SE similar by interchanging axes)

$$\begin{aligned}
 P'_{NE}(x, y) &= P\left(\frac{1}{x}, \frac{1}{y}\right) x^{d_x} y^{d_y} && \text{restricted to } 0 < x, y \leq \frac{1}{\bar{x}} \\
 P'_{NW}(x, y) &= P\left(x, \frac{1}{y}\right) y^{d_y} && \text{restricted to } 0 \leq x \leq \bar{x}, 0 < y \leq \frac{1}{\bar{x}} \\
 P'_{SE}(x, y) &= P\left(\frac{1}{x}, y\right) x^{d_x} && \text{restricted to } 0 < x \leq \frac{1}{\bar{x}}, 0 \leq y \leq \bar{x} \\
 P'_{SW}(x, y) &= P(x, y) && \text{restricted to } 0 \leq x \leq \bar{x}, 0 < y \leq \frac{1}{\bar{x}}
 \end{aligned}
 \tag{2.17}$$

Along the lines of Proposition 2.3.2 we can guarantee positivity of $P(x, y)$ given

positivity of the related polynomials (2.17).

Proposition 2.3.3. *Let $P(x, y)$ be a polynomial, $d_x = \deg_x(P)$, $d_y = \deg_y(P)$, and $\bar{x} > 0$. Consider the polynomials $P'_{NE}, P'_{SW}, P'_{NW}, P'_{SE}$ as defined in (2.17). If any one of these polynomials, generally denoted P'_\square , is positive on the region indicated in (2.17), then $P(x, y)$ is positive on the region \square . For example, if $P'_{NW}(x, y) \geq 0$ on $0 \leq x \leq \bar{x}$ and $0 < y \leq \frac{1}{\bar{x}}$, then $P(x, y) \geq 0$ on the region NW , and similarly for the other polynomials/regions.*

Proof. We will only see the proof for $P'_{NW}(x, y)$, the rest will follow in nearly the same manner. Assume that $P'_{NW}(x, y) \geq 0$ for $0 \leq x \leq \bar{x}$ and $0 < y \leq \frac{1}{\bar{x}}$. Then by definition of $P'_{NW}(x, y)$ we know that

$$P\left(x, \frac{1}{y}\right) y^{d_y} \geq 0 \quad \text{for } 0 \leq x \leq \bar{x}, 0 < y \leq \frac{1}{\bar{x}}.$$

Since y is strictly positive in the region in which $P'_{NW}(x, y)$ is defined, we can cancel y^{d_y} without reversing the inequality just as we did in the proof of Proposition 2.3.2. Now, let $y' := \frac{1}{y}$ to see that

$$P(x, y') \geq 0$$

for $0 \leq x \leq \bar{x}$ and $y' = \frac{1}{y} \geq \bar{x}$, which is precisely the region NW . The other regions will follow by doing substitutions $x' := \frac{1}{x}$ and $y' := \frac{1}{y}$ as necessary. Note that no work needs to be done for SW since $P'_{SW}(x, y) = P(x, y)$ and is defined in the region SW . \square

Next, we need to see how to prove that $P'_\square(x, y) \geq 0$ on the desired region. We will do this by subdividing the domain of P'_\square into finitely many smaller rectangles. Then, for each smaller rectangle, $S = \{a \leq x \leq b, c \leq y \leq d\}$, we transform it to \mathbb{R}_+^2 creating a corresponding polynomial, $P''_S(x, y)$, and test criteria **PosCoeffs** and **SubPoly** to see whether this polynomial is positive. See Figure 2.7 for the transformation of a general rectangle, S , to \mathbb{R}_+^2 . Given this transformation, the polynomial, P''_S , is given by

$$\begin{aligned} P''_S(x, y) &= \sigma_{\frac{1}{b-a}, \frac{1}{d-c}} \left(P'_\square \left(\frac{1}{x} + a, \frac{1}{y} + c \right) x^{d_x} y^{d_y} \right) \\ &= P'_\square \left(\frac{1}{x + \frac{1}{b-a}} + a, \frac{1}{y + \frac{1}{d-c}} + c \right) \left(x + \frac{1}{b-a} \right)^{d_x} \left(y + \frac{1}{d-c} \right)^{d_y} \end{aligned} \quad (2.18)$$

where P'_\square is one of P'_{NE} , P'_{NW} , P'_{SE} , P'_{SW} , and $d'_z = \text{degree of } z \text{ in } P'_\square$ for $z = x, y$. Before we see the canonical subdivision algorithm let us see why this $P''_S(x, y)$ will give

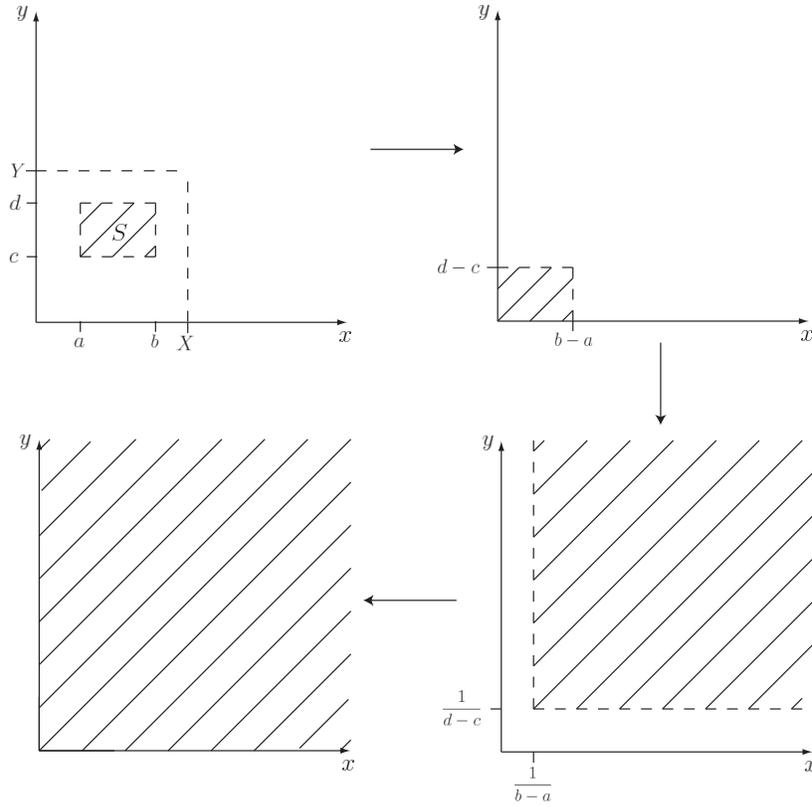


Figure 2.7: Transforming general rectangle, S , to \mathbb{R}_+^2

the desired result.

Proposition 2.3.4. *Let $P'_\square(x, y)$ be a polynomial, $d'_x = \deg_x(P'_\square)$, $d'_y = \deg_y(P'_\square)$, $0 \leq a < b$, and $0 \leq c < d$. Consider the polynomial P''_S as defined in (2.18). If $P''_S(x, y)$ is positive on \mathbb{R}_+^2 , then $P'_\square(x, y)$ is positive on the rectangle $S = \{a \leq x \leq b, c \leq y \leq d\}$.*

Proof. This proof follows the form of the proofs for Propositions 2.3.2 and 2.3.3. First, by definition of $P''(S)$, the fact that $P''_S(x, y) \geq 0$ for $x, y \geq 0$ means

$$P'_\square \left(\frac{1}{x + \frac{1}{b-a}} + a, \frac{1}{y + \frac{1}{d-c}} + c \right) \left(x + \frac{1}{b-a} \right)^{d'_x} \left(y + \frac{1}{d-c} \right)^{d'_y} \geq 0.$$

As in the previous proofs we may cancel $\left(x + \frac{1}{b-a} \right)^{d'_x} \left(y + \frac{1}{d-c} \right)^{d'_y}$ without reversing

the inequality. Let $x' = x + \frac{1}{b-a}$ and $y' = y + \frac{1}{d-c}$, then

$$P'_{\square} \left(\frac{1}{x'} + a, \frac{1}{y'} + c \right) \geq 0$$

for $x' = x + \frac{1}{b-a} \geq \frac{1}{b-a}$ and $y' = y + \frac{1}{d-c} \geq \frac{1}{d-c}$. Next, let $x'' = \frac{1}{x'}$ and $y'' = \frac{1}{y'}$. Making this substitution yields

$$P'_{\square} (x'' + a, y'' + c) \geq 0$$

for $0 < x'' = \frac{1}{x'} \leq b - a$ and $0 < y'' = \frac{1}{y'} \leq d - c$. For the final substitution, let $x''' = x'' + a$ and $y''' = y'' + c$. Then

$$P'_{\square} (x''', y''') \geq 0$$

for $a < x''' = x'' + a \leq b$ and $c < y''' = y'' + c \leq d$. In other words, $P'_{\square}(x, y) \geq 0$ for $(x, y) \in S$. \square

In principle any subdivision will work so long as we cover the entire finite rectangle. However, since the goal is to program the algorithm we need to specify a canonical subdivision. First we will simply divide into four equal regions. For each region we perform the above steps (transform the region and polynomial, apply criteria **PosCoeffs** and **SubPoly**). If we fail either criteria on a specific subregion, then we subdivide that subregion into four again and repeat. We continue to do this until we pass **PosCoeffs** or **SubPoly**, fail **LCoeff** or **Const** (and output **false**), or we reach some stopping condition and output **FAIL**. A stopping condition could be that we have subdivided N times, for some large N .

Before we summarize the positivity algorithm in the general n -dimensional case we must see the general **SubPoly** criteria. It was previously stated only in the case of 2 variables.

SubPoly- n : If the only negative coefficients in P_{\square} (including the constant term) are on terms of the form $x_r x_s$ then we check whether the quadratic form [25],

$$\sum_{\substack{i,j=1 \\ i \leq j}}^n a_{i,j} x_i x_j \tag{2.19}$$

where $a_{i,j}$ are coefficients of their respective terms in P_{\square} , is positive definite (i.e., is positive for all $\langle x_1, \dots, x_n \rangle \neq \langle 0, \dots, 0 \rangle$) using its corresponding matrix. The symmetric coefficient matrix is defined as: $A = (a'_{i,j})$, where $a'_{i,j} = a'_{j,i} = \frac{1}{2}a_{i,j}$ for all $i \neq j$, and $a'_{i,i} = a_{i,i}$ [30]. Given this matrix, we can equivalently think of the quadratic form as $\mathbf{x}A\mathbf{x}^T$, where $\mathbf{x} = \langle x_1, \dots, x_n \rangle$. Since A is a symmetric matrix, we know from the spectral theorem that it is diagonalizable by an orthonormal matrix, Q , so we have that $A = QDQ^T$ where D is a diagonal matrix. We can then rewrite the quadratic form as:

$$\begin{aligned}\mathbf{x}A\mathbf{x}^T &= \mathbf{x}QDQ^T\mathbf{x}^T \\ &= \tilde{\mathbf{x}}D\tilde{\mathbf{x}}^T.\end{aligned}$$

From this we easily see that the quadratic form is positive definite iff all eigenvalues of A are positive (i.e., A is positive definite). Then, if this quadratic “sub-polynomial” of P_{\square} is positive, P_{\square} itself is positive (since the other coefficients are positive).

We are now ready to summarize the algorithm in the n -dimensional case. Assume we have a polynomial $P \in \mathbb{R}[x_1, \dots, x_n]$, and want to test whether $P(x_1, \dots, x_n) \geq 0$ for $(x_1, \dots, x_n) \in \mathbb{R}_+^n$.

1. First cut \mathbb{R}_+^n into regions, similar to the NW , NE , SW , SE regions. For each variable we have 2 possibilities for its domain

$$0 \leq x_i \leq \bar{x} \quad \text{or} \quad \bar{x} \leq x_i < \infty.$$

A region is defined by making a choice for each variable, thus we have 2^n regions. The associated polynomial, P_{\square} , for each region is then created by substituting

$$x_i \text{ by } \begin{cases} x_i + \bar{x} & \text{if } \bar{x} \leq x_i < \infty \\ \frac{1}{x_i + \frac{1}{\bar{x}}} & \text{if } 0 \leq x_i \leq \bar{x} \end{cases}$$

in P , and then multiplying by $(x_i + \frac{1}{\bar{x}})^{d_{x_i}}$ if $0 \leq x_i \leq \bar{x}$.

2. For each region we check our 4 criteria **PosCoeffs**, **SubPoly- n** , **LCoeff**, and **Const**. If all 2^n polynomials pass **PosCoeffs** or **SubPoly- n** then we are done,

and P is positive on \mathbb{R}_+^n . If any of the polynomials fail **LCoeff** or **Const** then we are also done because we know that there are values in \mathbb{R}_+^n for the variables which make P negative. Otherwise, we continue on to step 3 for the regions which fail **PosCoeffs** and **SubPoly- n** .

3. Assume we have a specific region (domains for each variable), R , which failed step 2. Then we create the polynomial P'_R by substituting $\frac{1}{x_i}$ for x_i in P if x_i is restricted to $\bar{x} \leq x_i \leq \infty$ in R , and then multiplying by $x_i^{d_{x_i}}$ for those variables which were substituted. This new polynomial will be restricted to the region R' , which is defined from R in the following manner: if $0 \leq x_i \leq \bar{x}$ in R , then x_i has the same restriction in R' ; otherwise, $\bar{x} \leq x_i < \infty$ in R , and then x_i is restricted to $0 < x_i \leq \frac{1}{\bar{x}}$ in R' . More formally, if $D = \{i : 0 \leq x_i \leq \bar{x} \text{ in } R\}$, and $\bar{D} = [n] \setminus D$ then

$$R' = \left(\prod_{i \in D} \{0 \leq x_i \leq \bar{x}\} \right) \times \left(\prod_{i \in \bar{D}} \left\{ 0 < x_i \leq \frac{1}{\bar{x}} \right\} \right)$$

- (a) Subdivide R' into 2^n equal regions, S_j , and for each region create the polynomial P'_{S_j} in the same manner as (2.18).
- (b) Test positivity of P'_{S_j} using criteria **PosCoeffs**, **SubPoly- n** , **LCoeff**, and **Const**. If P'_{S_j} passes **PosCoeffs** or **SubPoly- n** then we are done in region S_j and can continue checking the rest of the subregions of R' . If P'_{S_j} fails **LCoeff** or **Const** then we stop altogether because we know that there are values for the variables in \mathbb{R}_+^n which make P negative. Otherwise, go back to step 3(a) with region R' now replaced by S_j .
- (c) If we have recursed more than N times (for some choice of N), stop and output **FAIL**.

Before going on to the proof-of-concept for a specific difference equation and equilibrium let us see how to apply the polynomial positivity algorithm. We will see two examples, one in which **SubPoly** must be used, and one where subdivisions are necessary.

Example 2.3.2. Let $P(x, y) = x^2 - xy + y^2$ and $\bar{x} = 1$. First we subdivide \mathbb{R}_+^2 into

the for regions NE , SW , NW , SE and get the following polynomials:

$$\begin{aligned} P_{NE}(x, y) &= \sigma_1(P) = x^2 - xy + y^2 + x + y + 1, \\ P_{SW}(x, y) &= \sigma_1\left(P\left(\frac{1}{x}, \frac{1}{y}\right) x^{d_x} y^{d_y}\right) = x^2 - xy + y^2 + x + y + 1, \\ P_{NW}(x, y) &= \sigma_{1,1}\left(P\left(\frac{1}{x}, y\right) x^{d_x}\right) = x^2 y^2 + 2x^2 y + 2xy^2 + x^2 + 3xy + y^2 + x + y + 1, \\ P_{SE}(x, y) &= \sigma_{1,1}\left(P\left(x, \frac{1}{y}\right) y^{d_y}\right) = x^2 y^2 + 2x^2 y + 2xy^2 + x^2 + 3xy + y^2 + x + y + 1. \end{aligned}$$

The polynomials $P_{NW}(x, y) = P_{SE}(x, y)$ have all positive coefficients, so they satisfy **PosCoeffs**. To see that $P_{NE}(x, y)$ (which equals $P_{SW}(x, y)$ in this example) is positive we must test criteria **SubPoly**.

The only negative coefficient in $P_{NE}(x, y)$ is on the xy term, so we look at the sub-polynomial $x^2 - xy + y^2$. The discriminant of this binary quadratic form is $4 \cdot 1 \cdot 1 - 1^2 = 3$, which is positive as needed. So we see that $P_{NE}(x, y)$ (and thus $P_{SW}(x, y)$) is positive by **SubPoly**.

In this example we don't have to do further subdivisions since $P_{NW}, P_{NE}, P_{SE}, P_{SW}$ are all positive. Therefore, we are done by Proposition 2.3.2. \diamond

Example 2.3.3. Let $P(x, y) = x^4 y - 5x^3 y + 10x^2 y + x + y$ and $\bar{x} = 1$. First we subdivide \mathbb{R}_+^2 into the for regions NE , SW , NW , SE and get the following polynomials:

$$\begin{aligned} P_{NE}(x, y) &= \sigma_1(P) = x^4 y + x^4 - x^3 y - x^3 + x^2 y + 2x^2 + 9xy + 11x + 7y + 8, \\ P_{SW}(x, y) &= \sigma_1\left(P\left(\frac{1}{x}, \frac{1}{y}\right) x^{d_x} y^{d_y}\right) = x^4 + 4x^3 + x^2 y + 17x^2 + 2xy + 21x + y + 8, \\ P_{NW}(x, y) &= \sigma_{1,1}\left(P\left(\frac{1}{x}, y\right) x^{d_x}\right) \\ &= x^4 y + x^4 + 4x^3 y + 4x^3 + 16x^2 y + 17x^2 + 19xy + 21x + 7y + 8, \\ P_{SE}(x, y) &= \sigma_{1,1}\left(P\left(x, \frac{1}{y}\right) y^{d_y}\right) = x^4 - x^3 + x^2 y + 2x^2 + 2xy + 11x + y + 8. \end{aligned}$$

In this example we see that $P_{SW}(x, y)$ and $P_{NW}(x, y)$ pass criteria **PosCoeffs** since all coefficients are positive. For the other two regions we will need to subdivide because the negative coefficients are not on the term xy .

Let us first examine SE . We need to create the polynomial $P'_{SE}(x, y)$ as in 2.17:

$$\begin{aligned} P'_{SE}(x, y) &= P\left(\frac{1}{x}, y\right) x^{d_x} \\ &= x^4 y + 10x^2 y + x^2 - 5xy + y \quad \text{restricted to } 0 < x \leq 1, 0 \leq y \leq 1. \end{aligned}$$

Then we subdivide the region $0 < x \leq 1, 0 \leq y \leq 1$ into four equal rectangles:

$$\begin{aligned} S_1 &= \left\{0 < x \leq \frac{1}{2}, 0 \leq y \leq \frac{1}{2}\right\}, & S_2 &= \left\{0 < x \leq \frac{1}{2}, \frac{1}{2} \leq y \leq 1\right\}, \\ S_3 &= \left\{\frac{1}{2} \leq x \leq 1, 0 \leq y \leq \frac{1}{2}\right\}, & S_4 &= \left\{\frac{1}{2} \leq x \leq 1, \frac{1}{2} \leq y \leq 1\right\}, \end{aligned}$$

and create four associated polynomials using (2.18):

$$\begin{aligned} P''_{S_1}(x, y) &= x^4 + 3x^3 + x^2 y + 6x^2 + 4xy + 20x + 4y + 25, \\ P''_{S_2}(x, y) &= \frac{1}{2}x^4 y + 2x^4 + \frac{3}{2}x^3 y + 6x^3 + 3x^2 y + 10x^2 + \\ &\quad + 10xy + 32x + \frac{25}{2}y + 42, \\ P''_{S_3}(x, y) &= \frac{1}{4}x^4 y + \frac{25}{16}x^4 + 3x^3 y + 20x^3 + 13x^2 y + \\ &\quad + 96x^2 + 24xy + 196x + 16y + 144, \\ P''_{S_4}(x, y) &= \frac{25}{32}x^4 y + \frac{21}{8}x^4 + 10x^3 y + 34x^3 + 48x^2 y + \\ &\quad + 166x^2 + 98xy + 344x + 72y + 256. \end{aligned}$$

All four polynomials for the subdivision of SE are positive by **PosCoeffs**, therefore $P'_{SE}(x, y) \geq 0$, and by Proposition 2.3.3 we see that $P(x, y) \geq 0$ on the region SE .

Now let us look at P_{NE} . Create P'_{NE} as indicated by (2.17):

$$\begin{aligned} P'_{NE}(x, y) &= P\left(\frac{1}{x}, \frac{1}{y}\right) x^{d_x} y^{d_y} \\ &= 8x^4 y + 7x^4 + 11x^3 y + 9x^3 + 2x^2 y + x^2 - xy - x + y + 1 \\ &\quad \text{restricted to } 0 < x \leq 1, 0 \leq y \leq 1. \end{aligned}$$

Subdivide the region $0 < x \leq 1, 0 \leq y \leq 1$ into the same $S_1, S_2, S_3,$ and S_4 as above,

and create the polynomials P''_{S_i} , this time from P'_{NE} :

$$\begin{aligned}
P''_{S_1}(x, y) &= x^4y + 3x^4 + 7x^3y + 21x^3 + 19x^2y + 58x^2 + \\
&\quad + 33xy + 105x + 37y + 120, \\
P''_{S_2}(x, y) &= \frac{3}{2}x^4y + 4x^4 + \frac{21}{2}x^3y + 28x^3 + 29x^2y + 78x^2 + \\
&\quad + \frac{105}{2}xy + 144x + 60y + 166, \\
P''_{S_3}(x, y) &= \frac{37}{16}x^4y + \frac{15}{2}x^4 + \frac{115}{4}x^3y + \frac{375}{4}x^3 + 142x^2y + \\
&\quad + 463x^2 + 320xy + 1040x + 272y + 880, \\
P''_{S_4}(x, y) &= \frac{15}{4}x^4y + \frac{83}{8}x^4 + \frac{375}{8}x^3y + 130x^3 + \frac{463}{2}x^2y + \\
&\quad + 642x^2 + 520xy + 1440x + 440y + 1216.
\end{aligned}$$

As before, all four subdivision polynomials pass **PosCoeffs**, and so they are positive. Therefore, by Propositions 2.3.2, 2.3.3, and 2.3.4 we know that $P(x, y) \geq 0$ for $(x, y) \in \mathbb{R}_+^2$. \diamond

2.4 Proof of Concept

We have now seen the full algorithm to prove GAS of equilibrium points of rational difference equations. However, there is no reason a priori that this algorithm is applicable. It could be the case that no such K (see Section 2.2 for a definition of K) exists, and this algorithm would be useless.

We will now see that this technique does, in fact, work to prove global asymptotic stability of an equilibrium of a particular rational difference equation. The proof of the following theorem to establish global asymptotic stability will go through the procedure outlined in the previous sections.

Theorem 2.4.1. *For the rational difference equation*

$$x_{n+1} = \frac{4 + x_n}{1 + x_{n-1}}, \quad (2.20)$$

the equilibrium, $\bar{x} = 2$, is GAS.

Proof. We will prove that $K = 5$ satisfies (2.8) from Theorem 2.1.10. From the rational difference equation, the equilibrium $\bar{x} = 2$, and $K = 5$ we get the polynomial, $P := P_{(2,2),5}(\langle x_1, x_2 \rangle)$, as defined in (2.13):

$$\begin{aligned}
P = & 25x_1^8x_2^4 + 340x_1^8x_2^3 + 1606x_1^8x_2^2 + 3060x_1^8x_2 + 2025x_1^8 + 60x_1^7x_2^5 + 1158x_1^7x_2^4 + \\
& + 8460x_1^7x_2^3 + 28936x_1^7x_2^2 + 45848x_1^7x_2 + 27090x_1^7 + 71x_1^6x_2^6 + 1418x_1^6x_2^5 + \\
& + 11229x_1^6x_2^4 + 53362x_1^6x_2^3 + 147345x_1^6x_2^2 + 207144x_1^6x_2 + 113103x_1^6 + 72x_1^5x_2^7 + \\
& + 1420x_1^5x_2^6 + 9012x_1^5x_2^5 + 20174x_1^5x_2^4 + 24716x_1^5x_2^3 + 74718x_1^5x_2^2 + 163032x_1^5x_2 + \\
& + 108952x_1^5 + 47x_1^4x_2^8 + 1276x_1^4x_2^7 + 11120x_1^4x_2^6 + 25528x_1^4x_2^5 - 118780x_1^4x_2^4 - \\
& - 688300x_1^4x_2^3 - 1195361x_1^4x_2^2 - 790736x_1^4x_2 - 148969x_1^4 + 12x_1^3x_2^9 + 538x_1^3x_2^8 + \\
& + 7854x_1^3x_2^7 + 45864x_1^3x_2^6 + 53604x_1^3x_2^5 - 515564x_1^3x_2^4 - 2066454x_1^3x_2^3 - \\
& - 2469564x_1^3x_2^2 - 207576x_1^3x_2 + 833882x_1^3 + x_1^2x_2^{10} + 86x_1^2x_2^9 + 2109x_1^2x_2^8 + \\
& + 22070x_1^2x_2^7 + 102117x_1^2x_2^6 + 105526x_1^2x_2^5 - 695269x_1^2x_2^4 - 1867364x_1^2x_2^3 + \\
& + 785343x_1^2x_2^2 + 6256056x_1^2x_2 + 4716817x_1^2 + 4x_1x_2^{10} + 198x_1x_2^9 + 3530x_1x_2^8 + \\
& + 29636x_1x_2^7 + 117218x_1x_2^6 + 136288x_1x_2^5 - 289440x_1x_2^4 + 253318x_1x_2^3 + \\
& + 5674806x_1x_2^2 + 11634024x_1x_2 + 7054300x_1 + 4x_2^{10} + 148x_2^9 + 2145x_2^8 + 15348x_2^7 + \\
& + 53870x_2^6 + 69340x_2^5 + 30579x_2^4 + 801874x_2^3 + 3802411x_2^2 + 6262908x_2 + 3488704.
\end{aligned}$$

The goal is to prove that this polynomial is positive when all variables are positive. Recall that we created this polynomial by taking the numerator of

$$\|\mathcal{X} - \bar{\mathcal{X}}\|^2 - \|Q^5(\mathcal{X}) - \bar{\mathcal{X}}\|^2,$$

where $Q(\mathcal{X})$ is the map

$$Q\left(\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}\right) = \begin{bmatrix} \frac{4+x_n}{1+x_{n-1}} \\ x_n \end{bmatrix}.$$

Now we run the polynomial positivity algorithm described in Section 2.3 to prove that this polynomial is positive. If the polynomial is positive when all variables are positive then the equilibrium, $\bar{x} = 2$, is GAS for the original difference equation by Theorem 2.1.10.

First we will prove that $P > 0$ in the region NE . We make the polynomial P_{NE} by substituting $x_1 = x_1 + 2$ and $x_2 = x_2 + 2$ into P . See Appendix A.3 for the polynomial P_{NE} . Now we need to prove that $P_{NE} > 0$ in the region \mathbb{R}_+^2 except when all variables are simultaneously zero. The only negative coefficient is on the term x_1x_2 , so we can use the discriminant method. The binary quadratic form that we must show is positive definite is

$$349366689x_1^2 - 6980904x_1x_2 + 318700575x_2^2.$$

Its discriminant is $d = 445324725659927484$ which is positive, so by **SubPoly** $P_{NE} > 0$ in \mathbb{R}_+^2 .

Now we will prove $P > 0$ in the region NW . Create the polynomial P_{NW} by substituting $x_1 = 1/x_1$, multiplying by $x_1^{d_{x_1}} = x_1^8$, and then translating x_1 by $1/2$ to the left, and x_2 by 2 to the left. See Appendix A.3 for the polynomial P_{NW} . All coefficients in P_{NW} are positive, and the constant term is zero. There is no proper subset of the variables for which setting them all equal zero yields the zero polynomial. Therefore, P_{NW} is zero only when all variables are zero, and so $P > 0$ in NW .

Next, we will prove $P > 0$ in the region SE . First make the polynomial P_{SE} by substituting $x_2 = 1/x_2$, multiplying by $x_2^{d_{x_2}} = x_2^{10}$, and then translating x_1 by 2 to the left and x_2 by $1/2$ to the left. See Appendix A.3 for the polynomial P_{SE} . Now we need to prove that $P_{SE} > 0$ in the region \mathbb{R}_+^2 except when all variables are simultaneously zero. All coefficients are positive, and the constant term is zero. There is no proper subset of the variables for which setting them all equal zero yields the zero polynomial. Therefore, P_{SE} is zero only when all variables are zero, and then $P > 0$ in the region SE .

Finally, we must prove $P > 0$ in the region SW . Make the polynomial P_{SW} by substituting $x_1 = 1/x_1$ and $x_2 = 1/x_2$, multiplying by $x_1^{d_{x_1}} = x_1^8$ and $x_2^{d_{x_2}} = x_2^{10}$, and then translating both variables by $1/2$ to the left. See Appendix A.3 for the polynomial P_{SW} . Now we need to prove that $P_{SW} > 0$ in the region \mathbb{R}_+^2 except when all variables are simultaneously zero. As in the region NE the term x_1x_2 has a negative coefficient (and that is the only such coefficient), so we will use the discriminant method again.

The binary quadratic form that must be positive is

$$\frac{349366689}{16384}x_1^2 - \frac{872613}{2048}x_1x_2 + \frac{318700575}{16384}x_2^2$$

The discriminant is $d = \frac{111331181414981871}{67108864}$, which is positive. Then, by **SubPoly**, P_{SW} is positive in \mathbb{R}_+^2 , so $P > 0$ in the region SW .

Since $P > 0$ in all four regions, NE , NW , SE , and SW , the K value 5 is proven to work for the rational difference equation $x_{n+1} = \frac{4+x_n}{1+x_{n-1}}$ \square

We can now see that the algorithm is indeed applicable. However, it wouldn't be possible without programming the algorithm. For large K values, even $K \geq 3$, the polynomials are near impossible to deal with by hand. I have created a Maple package to conjecture K values and prove global asymptotic stability. For a description of the most useful procedures in the maple code see Section 2.6.

2.5 Results

In this section we will present the results that my algorithm can prove in full generality. For a list in the spirit of Elias Camouzis and Gerasimos Ladas in their book [3] see Appendix A.1. Also found in the appendix are some of the results that my algorithm can prove when values for the parameters are given, but cannot prove in as much generality as here. Typically this occurs when the K value is not uniform over the set of parameters as it is in Theorems 2.5.1 - 2.5.10.

Theorem 2.5.1. *The rational difference equation $x_{n+1} = \beta x_n$ is GAS iff $\beta < 1$.*

Proof. For this rational difference equation, the equilibrium is $\bar{x} = 0$ and it is LAS iff $\beta < 1$ by the Linearized Stability Theorem. To show that it is GAS when $\beta < 1$ we must prove that the polynomial, $P_{(0),1}(x_1) = 1 - \beta^2$, is positive. Clearly $P_{(0),1}(x_1)$ is only positive when $\beta < 1$. Therefore, $\bar{x} = 0$ is GAS iff $\beta < 1$. \square

Theorem 2.5.2. *The rational difference equation $x_{n+1} = \gamma x_{n-1}$ is GAS iff $\gamma < 1$.*

Proof. For this rational difference equation, the only non-negative equilibrium is $\bar{x} = 0$, and it is LAS iff $\gamma < 1$. To prove GAS for $\gamma < 1$ we must prove that the polynomial,

$P_{(0,0),1}(x_1, x_2) = x_2^2 - x_2^2\gamma^2$ is positive. There is a negative coefficient, so we will do a substitution to get the range of γ to be $(0, \infty)$. First let $\gamma = \frac{1}{\gamma_1}$ and multiply by γ_1^2 . Then $1 < \gamma_1 < \infty$. Next, let $\gamma_1 = \gamma_2 + 1$ so that $0 < \gamma_2 < \infty$. Now the polynomial is

$$\tilde{P} = x_2^2(\gamma_2^2 + 2\gamma_2)$$

which has all positive coefficients. Therefore, for $0 \leq \gamma < 1$ the equilibrium $\bar{x} = 0$ is GAS. \square

Theorem 2.5.3. *The rational difference equation $x_{n+1} = \frac{1}{4} \frac{M^2-1}{1+x_n}$ is GAS iff $|M| > 1$.*

Proof. For this rational difference equation there are two possible equilibrium points, $\bar{x} = \frac{1}{2}(M-1)$ and $\bar{x} = -\frac{1}{2}(M+1)$. We require that all coefficients, and the equilibrium point be positive. There are two cases for which the coefficient, $M^2-1 = (M-1)(M+1)$, is positive, each corresponds to one of the two equilibria:

$$\begin{aligned} M-1 > 0 \text{ and } M+1 > 0, \quad \bar{x} &= \frac{1}{2}(M-1), \\ M-1 < 0 \text{ and } M+1 < 0, \quad \bar{x} &= -\frac{1}{2}(M+1). \end{aligned}$$

In the first case, the polynomial, $P_{(\frac{1}{2}(M-1), 2)}(x)$, is

$$\begin{aligned} P_1 := & 64x^4 + (32M^2 - 64M + 160)x^3 + (-16M^3 + 48M^2 - 112M + 144)x^2 + \\ & - 8(M-1)(M^3 + M^2 - M + 7)x + 4(M+1)(M^2 - M + 2)(M-1)^2 \end{aligned}$$

and we require that $M > 1$. Substitute $M = M_1 + 1$ to get the following polynomial

$$\begin{aligned} \tilde{P}_1 = & 64x^4 + (32M_1^2 + 128)x^3 + (-16M_1^3 - 64M_1 + 64)x^2 + \\ & - 8M_1(M_1^3 + 4M_1^2 + 4M_1 + 8)x + 4M_1^2(M_1 + 2)(M_1^2 + M_1 + 2). \end{aligned}$$

Since not all coefficients here are positive, we must notice (or ask our computer algebra system) that \tilde{P}_1 factors.

$$\tilde{P}_1 = 4(M_1^2 + M_1 + 2 + 2x)(M_1 + 2 + 2x)(M_1 - 2x)^2$$

Since each term in the product is positive we know that \tilde{P}_1 is positive. So we see that when $M > 1$, $\bar{x} = \frac{1}{2}(M-1)$ is GAS.

In the second case, the polynomial $P_{\langle -\frac{1}{2}(M+1) \rangle, 2}(x)$ is

$$P_2 := 64x^4 + (32M^2 + 64M + 160)x^3 + (16M^3 + 48M^2 + 112M + 144)x^2 + \\ - 8(M+1)(M^3 - M^2 - M - 7)x - 4(M-1)(M^2 + M + 2)(M+1)^2$$

and we require that $M < -1$. Substitute $M = M_1 - 1$, and then $M_1 = -M_2$. Then the domain of M_2 is $(0, \infty)$ as needed. We then get the polynomial,

$$\tilde{P}_2 = 64x^4 + (32M_2^2 + 128)x^3 - (16M_2^3 + 64M_2 - 64)x^2 + \\ - 8M_2(M_2^3 + 4M_2^2 + 4M_2 + 8)x + 4M_2^2(M_2 + 2)(M_2^2 + M_2 + 2).$$

As before we must notice that \tilde{P}_2 factors into a product of positive terms and is therefore positive:

$$\tilde{P}_2 = 4(M_2^2 + M_2 + 2 + 2x)(M_2 + 2 + 2x)(M_2 - 2x)^2.$$

In both cases P is positive, so \bar{x} is GAS iff $|M| > 1$. □

Theorem 2.5.4. *The rational difference equation $x_{n+1} = \frac{\beta x_n}{1+x_n}$ is GAS for all positive $\beta \neq 1$.*

Proof. For this rational difference equation there are two equilibria, $\bar{x} = 0$ and $\bar{x} = \beta - 1$. If $0 \leq \beta < 1$, the only non-negative equilibrium is $\bar{x} = 0$ which is LAS. If $\beta > 1$ the equilibrium $\bar{x} = \beta - 1$ is LAS. If $\beta = 1$ then the equilibrium $\bar{x} = 0$ is not LAS.

First, assume $0 \leq \beta < 1$ and thus the equilibrium is $\bar{x} = 0$. The polynomial $P_{\langle 0 \rangle, 1}(x)$ is then

$$P_1 := x^2 + 2x + 1 - \beta^2.$$

The coefficients are positive, and the constant term is positive because $0 \leq \beta < 1$, so $\bar{x} = 0$ is GAS when $0 \leq \beta < 1$.

Now assume $\beta \geq 1$, and the equilibrium is $\bar{x} = \beta - 1$. The polynomial $P_{\langle \beta-1 \rangle, 1}(x)$ is

$$P_2 := x^4 + (4 - 2\beta)x^3 + (\beta^2 - 6\beta + 5)x^2 + (2\beta^2 - 4\beta + 2)x.$$

In this case the positivity method described in section 2.3 does not apply because there is no way to make the coefficient of x^3 positive using the tricks we have used in previous

proofs. However, we can still prove positivity of P_2 by noticing that it factors into a product of positive terms,

$$P_2 = x(x+2)(\beta-1-x)^2.$$

Therefore, since P_1 and P_2 are positive when $x > 0$ we have that the equilibria are GAS when $\beta > 0$ and $\beta \neq 1$. \square

Theorem 2.5.5. *The rational difference equation $x_{n+1} = \frac{x_{n-1}}{A+x_n}$ is GAS iff $A > 1$.*

Proof. For this rational difference equation there are two equilibrium points, $\bar{x} = 0$ and $\bar{x} = 1 - A$. If $0 \leq A < 1$ neither equilibrium is LAS. If $A > 1$ the only non-negative equilibrium is $\bar{x} = 0$ and it is LAS. To prove that it is GAS we must show that the following polynomial, $P_{(0,0),1}(x_1, x_2)$, is positive when $A > 1$,

$$P := x_1^2 x_2^2 + 2A x_1 x_2^2 + A^2 x_2^2 - x_2^2.$$

In order to get the range of A to be $(0, \infty)$ we must make the substitution $A = A_1 + 1$.

The resulting polynomial is

$$\tilde{P} = x_1^2 x_2^2 + 2A_1 x_1 x_2^2 + 2x_1 x_2^2 + A_1^2 x_2^2 + 2A_1 x_2^2.$$

Since all coefficients in \tilde{P} are positive when $A_1 \in (0, \infty)$, the polynomial is positive.

Therefore $\bar{x} = 0$ is GAS when $A > 1$. \square

Theorem 2.5.6. *The rational difference equation $x_{n+1} = \frac{x_{n-1}}{A+x_{n-1}}$ is GAS for all positive values of A .*

Proof. For this rational difference equation there are two possible equilibrium points, $\bar{x} = 0$ and $\bar{x} = 1 - A$. If $A > 1$ then the only non-negative equilibrium is $\bar{x} = 0$, and it is LAS. To prove that it is also GAS we must show that the following polynomial, $P_{(0,0),1}(x_1, x_2)$, is positive for $A > 1$,

$$P_1 := x_2^4 + 2A x_2^3 + A^2 x_2^2 - x_2^2.$$

We do the substitution $A = A_1 + 1$ to get the polynomial

$$\tilde{P}_1 = x_2^4 + 2A_1 x_2^3 + 2x_2^3 + A_1^2 x_2^2 + 2A_1 x_2^2.$$

All coefficients are now positive, so the polynomial is positive or 0 when all variables are positive. Therefore $\bar{x} = 0$ is GAS when $A > 1$.

If $0 < A < 1$ then the equilibrium $\bar{x} = 0$ is not LAS. However, the equilibrium $\bar{x} = 1 - A$ is LAS. The polynomial $P_{\langle 1-A, 1-A \rangle, 2}(x_1, x_2)$ is

$$P_2 := x_2^4 + (4A - 2)x_2^3 + (5A^2 - 6A + 1)x_2^2 + (2A^3 - 4A^2 + 2A)x_2$$

Since A is restricted between 0 and 1 we can do the substitution, $A = 1/A_1$ and multiply by A_1^3 , followed by $A_1 = A_2 + 1$. Then we are concerned with $A_2 \in (0, \infty)$. The polynomial becomes

$$\tilde{P}_2 = x_2(2 + x_2 + A_2x_2)(-A_2 + x_2 + A_2x_2)^2.$$

Since each of the terms is positive we see that $\bar{x} = 1 - A$ is GAS when $0 < A < 1$. \square

Theorem 2.5.7. *The rational difference equation $x_{n+1} = \alpha + \beta x_n$ is GAS iff $\beta < 1$.*

Proof. For this rational difference equation the equilibrium is $\bar{x} = \frac{\alpha}{1-\beta}$, which will be positive iff $\beta < 1$. For $K = 1$ the polynomial $P_{\langle \frac{\alpha}{1-\beta} \rangle, 1}(x)$ is

$$\begin{aligned} P &:= (\beta + 1)(1 - \beta)^5 x^2 - 2\alpha(\beta + 1)(1 - \beta)^4 x + \alpha^2(\beta + 1)(1 - \beta)^3 \\ &= (\beta + 1)(1 - \beta)^3 [(1 - \beta)^2 x^2 - 2\alpha(1 - \beta)x + \alpha^2] \\ &= (\beta + 1)(1 - \beta)^3 [(1 - \beta)x - \alpha] \end{aligned}$$

Since the domain of β is $[0, 1)$ we must make some substitutions in order for the parameter to have domain $(0, \infty)$. First substitute $\beta = \frac{1}{\beta_1}$ and multiply by β_1^6 . Next, make the substitution $\beta_1 = \beta_2 + 1$ to arrive at

$$\tilde{P} = \beta_2^3(2 + \beta_2)(-\beta_2x + \alpha\beta_2 + \alpha)^2$$

which is clearly positive. Therefore, the equilibrium $\bar{x} = \frac{\alpha}{1-\beta}$ is GAS when $\beta < 1$. \square

Theorem 2.5.8. *The rational difference equation $x_{n+1} = q + \frac{1}{4} \frac{M^2 - q^2}{x_n}$ is GAS iff $q > 0$ and $M^2 - q^2 > 0$.*

Proof. For this rational difference equation there are two possible equilibria

$$\bar{x} = \frac{1}{2}(M + q), \text{ and } \bar{x} = \frac{1}{2}(q - M).$$

We require that all coefficients, and the equilibrium point be positive, so there are two cases for which the coefficients, $M^2 - q^2 = (M - q)(M + q)$ and q , are positive. Each case corresponds to one of the two equilibrium points:

$$\begin{aligned} M - q < 0, M + q < 0, \text{ and } q > 0, \text{ with } \bar{x} &= -\frac{1}{2}(M - q), \\ M - q > 0, M + q > 0, \text{ and } q > 0, \text{ with } \bar{x} &= \frac{1}{2}(M + q). \end{aligned}$$

In the first case, we consider the region given by $\{0 < q, M < -q\}$. The polynomial $P_{\langle -\frac{1}{2}(M-q) \rangle, 2}(x)$ is

$$P_1 := -4q(Mq + 2qx + M^2)(-2x + M + q)(2x + M - q)^2.$$

Substitute $M = M_1 - q$ and then $M_1 = -M_2$ to get the following polynomial

$$\tilde{P}_1 = 4q(qM_2 + 2qx + M_2^2)(2x + M_2)(M_2 + 2q - 2x)^2.$$

Now the region that the polynomial must be positive in is $\{M_2 > 0, q > 0, x > 0\}$. Since \tilde{P}_1 is a product of positive polynomials (either all coefficients are positive or it is squared), we know that \tilde{P}_1 is positive.

In the second case, the region is $\{0 < q, q < M\}$. The polynomial $P_{\langle \frac{1}{2}(M+q) \rangle, 2}(x)$ is

$$P_2 := 4q(M^2 - Mq + 2qx)(M - q + 2x)(M + q - 2x)^2.$$

Substitute $M = M_1 + q$ to get the polynomial

$$\tilde{P}_2 = 4q(M_1^2 + M_1q + 2qx)(M_1 + 2x)(M_1 + 2q - 2x)^2.$$

As before, \tilde{P}_2 is positive. Therefore, the equilibrium of $x_{n+1} = q + \frac{1}{4} \frac{M^2 - q^2}{x_n}$ is GAS iff $q > 0$ and $|M| > q$. \square

Theorem 2.5.9. *The rational difference equation $x_{n+1} = \frac{1}{4} \frac{M^2 - q^2 + 4x_n}{1 + q + x_n}$ is GAS iff*

$q > -1$ and $M^2 - q^2 > 0$.

Proof. For the rational difference equation there are two equilibria,

$$\bar{x} = \frac{1}{2}(M - q) \text{ and } \bar{x} = -\frac{1}{2}(M + q).$$

This is the most complicated of all the order 1 rational difference equations, with linear numerator and denominator, because of the possibilities for the parameters. We require that all coefficients and the equilibrium are positive. In order for the coefficients to be positive we must have

$$\begin{aligned} M^2 - q^2 > 0 \text{ and } 1 + q > 0 \\ (M - q)(M + q) > 0 \text{ and } q > -1. \end{aligned}$$

Therefore,

$$\{M - q > 0 \text{ and } M + q > 0\} \text{ and } q > -1,$$

or

$$\{M - q < 0 \text{ and } M + q < 0\} \text{ and } q > -1.$$

Now, if we are in the first case then the positive equilibrium is $\bar{x} = \frac{1}{2}(M - q)$ since $M - q > 0$. If we are in the second case then the positive equilibrium is $\bar{x} = -\frac{1}{2}(M + q)$ since $M + q < 0$.

Let's look at the first case. Another way to state it is

$$\{1 < M \text{ and } -1 < q < M\} \text{ or } \{0 < M \leq 1 \text{ and } -M < q < M\}$$

with equilibrium $\bar{x} = \frac{1}{2}(M - q)$. The polynomial, $P_{\langle \frac{1}{2}(M-q) \rangle, 2}(x)$ is

$$P_1 := 4(q + 2)(M^2 - Mq + 2q^2 - 2M + 6q + 4 + 2qx + 4x)(M + q + 2x)(M - q - 2x)^2.$$

First, consider the case where $\{1 < M \text{ and } -1 < q < M\}$. Then we must substitute $M = M_1 + 1$ and $q = q_1 - 1$ into P_1 to yield the following polynomial in which $M_1 > 0$ and $0 < q_1 < M_1 + 2$.

$$\begin{aligned} \tilde{P}_{1,1} &= 4(1 + q_1)(M_1^2 + M_1 - M_1q_1 + q_1 + 2q_1^2 + 2x + 2xq_1) \cdot \\ &\quad \cdot (2x + M_1 + q_1)(-2x + M_1 + 2 - q_1)^2. \end{aligned}$$

Since we still have negative terms, and q_1 has a finite interval as its domain, we make the substitution $q_1 = \frac{1}{q_2}$ and multiply by q_2^6 , followed by the substitution $q_2 = q_3 + \frac{1}{M_1+2}$.

$$\begin{aligned} \tilde{P}_{1,1} = \frac{4}{(M_1+2)^6} & \left[((M_1+2)^2 q_3 - 2x(M_1 q_3 + 2q_3 + 1))^2 (M_1 q_3 + M_1 + 2q_3 + 3) \cdot \right. \\ & \cdot (M_1^2 q_3 + 2M_1 q_3 + 2M_1 + 2 + 2x(M_1 q_3 + 2q_3 + 1)) \cdot \\ & \cdot \left(2x(M_1 q_3 + 2q_3 + 1)(M_1 q_3 + M_1 + 2q_3 + 3) + \right. \\ & + M_1^4 q_3^2 + 5M_1^3 q_3^2 + M_1^3 q_3 + 8M_1^2 q_3^2 + 3M_1^2 q_3 + \\ & \left. \left. + 4M_1 q_3^2 + 2M_1^2 + 4M_1 q_3 + 8M_1 + 4q_3 + 10 \right) \right]. \end{aligned}$$

Notice that each (multiplicative) term of $\tilde{P}_{1,1}$ is positive, either it is squared or it has no negative coefficients. Therefore $\tilde{P}_{1,1}$ is positive.

Now let $\{0 < M \leq 1 \text{ and } -M < q < M\}$ and perform the substitutions

$$\begin{aligned} M &= \frac{1}{M_1}, \text{ and then multiply by } M_1^3, \\ M_1 &= M_2 + 1, \\ q &= q_1 - \frac{1}{M_2 + 1}. \end{aligned}$$

The region is then $\{0 \leq M_2, 0 < q_1 < \frac{2}{M_2+1}\}$, and the polynomial is

$$\begin{aligned} \tilde{P}_{1,2} = 4(M_1+2) & \left[(q_1 + 2x)^2 (M_2 q_1 + 2M_2 + q_1 + 3)(M_2 q_1 + q_1 + 2 + 2xM_2 + 2x) \cdot \right. \\ & \cdot \left(2(M_2 + 1)(q_1 M_2 + 2M_2 + q_1 + 3) + 2M_2^2 q_1^2 + 6M_2^2 q_1 + 4M_2 q_1^2 + \right. \\ & \left. \left. + 4M_2^2 + 15M_2 q_1 + 2q_1^2 + 12M_2 + 9q_1 + 10 \right) \right]. \end{aligned}$$

Since all variables are positive, and in this factored polynomial all coefficients are positive, the polynomial is positive as needed.

Now we must do the second case, where the equilibrium is $-\frac{1}{2}(M+q)$, and the parameters satisfy $\{M-q < 0 \text{ and } M+q < 0\}$ and $q > -1$. Again, we can restate this as

$$\{M < -1 \text{ and } -1 < q < -M\} \text{ or } \{-1 \leq M < 0 \text{ and } M < q < -M\}.$$

The polynomial $P_{\langle -\frac{1}{2}(M+q), 2 \rangle}(x)$ is

$$P_2 := 4(2+q)(M^2 + Mq + 2q^2 + 2M + 6q + 4 + 2x(2+q)) \cdot (2x - M + q)(2x + M + q)^2.$$

In the first sub-case we assume that $\{M < -1 \text{ and } -1 < q < -M\}$. In order to transform the region into one which all variables are positive we must do the substitutions

$$\begin{aligned} M &= M_1 - 1, \\ q &= q_1 - 1, \\ M_1 &= -M_2. \end{aligned}$$

Then the region is $\{0 < M_2 \text{ and } 0 < q_1 < M_2 + 2\}$, so all variables are positive, and the polynomial is

$$\begin{aligned} \tilde{P}_{2,1} &= 4(1+q_1)(M_2 + q_1 + 2x)(M_2 - q_1 + 2 - 2x)^2 \cdot (M_2^2 - M_2q_1 + 2q_1^2 + M_2 + q_1 + 2x(1+q_1)). \end{aligned}$$

You may notice that there is a negative coefficient in one of these terms, namely in the term $M_2^2 - M_2q_1 + 2q_1^2 + M_2 + q_1 + 2x(1+q_1)$. However, we can show that this term is still positive using the **SubPoly** method. The negative coefficient is on the term M_2q_1 . The sub-polynomial is $M_2^2 - M_2q_1 + 2q_1^2$ whose binary quadratic form discriminant is $4 \cdot 1 \cdot 1 - (-1)^2 = 3 > 0$. Therefore, this term is positive, and so $\tilde{P}_{2,1} > 0$ as well.

For the second sub-case we assume that $\{-1 \leq M < 0 \text{ and } M < q < -M\}$. Again we transform the region into one where all variables are positive using the following substitutions

$$\begin{aligned} M &= -M_1, \\ q &= q_1 + M_1, \\ M_1 &= \frac{1}{M_2}, \text{ and then multiply by } M_2^3, \\ M_2 &= M_3 + 1. \end{aligned}$$

Then the region becomes $\{0 \leq M_3, 0 < q_1 < \frac{2}{M_3+1}\}$, and the polynomial is

$$\begin{aligned} \tilde{P}_{2,2} = & 4(2x + q_1)(M_3q_1 + 2M_3 + q_1 + 1)(M_3q_1 + q_1 - 2 + 2xM_3 + 2x)^2 \\ & \cdot \left(2M_3^2q_1^2 + 6M_3^2q_1 + 4M_3q_1^2 + 4M_3^2 + 7M_3q_1 + 2q_1^2 + q_1 + \right. \\ & \left. + 2x(M_3 + 1)(M_3q_1 + 2M_3 + q_1 + 1) \right). \end{aligned}$$

Again, all terms are positive, and so the polynomial is positive. That is the final case for this rational difference equation, so the equilibrium is GAS iff $q > -1$ and $M^2 - q^2 > 0$. \square

Theorem 2.5.10. *The rational difference equation $x_{n+1} = \frac{x_{n-1}}{A+Bx_n+x_{n-1}}$ is GAS iff $A > 1$.*

Proof. For this rational difference equation there are two possible equilibrium points, $\bar{x} = 0$ and $\bar{x} = \frac{1-A}{B+1}$. If $0 < A < 1$ neither equilibrium is LAS. If $A > 1$ the only positive equilibrium is $\bar{x} = 0$, and it is LAS. In order to show that it is GAS we must prove that the following polynomial, $P_{(0,0),1}(x_1, x_2)$,

$$P := x_2^4 + 2Bx_2^3x_1 + B^2x_2^2x_1^2 + 2Ax_2^3 + 2ABx_2^2x_1 + (A^2 - 1)x_2^2,$$

is positive when $A > 1$ and $B, x_1, x_2 > 0$. In order to get all variables in the range $(0, \infty)$ we do the substitution $A = A_1 + 1$ and get

$$\tilde{P} = x_2^4 + 2Bx_2^3x_1 + B^2x_2^2x_1^2 + (2A_1 + 2)x_2^3 + (2BA_1 + 2B)x_2^2x_1 + (A_1^2 + 2A_1)x_2^2.$$

Since all coefficients are positive, the polynomial is positive when all variables and parameters are in the range $(0, \infty)$. Therefore, $\bar{x} = 0$ is GAS when $A > 1$. \square

2.6 Maple Code

The Maple package that I have written to accompany this chapter contains many procedures. There is a Help function (type `Help()` to see a list of all procedures, and `Help(<<procedure name>>)` to get help on a specific procedure). The following is a list of the most common inputs for the procedures.

R - a rational difference equation

vars - variables in R

K - K value to test, see equation 2.8 for criteria that K satisfies

MinK - minimum K value to test

MaxK - maximum K value to test

N - positive integer, the number of times to subdivide the finite regions, see step 3 in the algorithm at the end of section 2.3

Here is a description of the most useful procedures in the code:

ProveK: A procedure to prove that a specific K value works for a specific rational difference equation.

Inputs: R, vars, K, N

Output: List of lists of length 2. The first element of each inner list is the equilibrium, the second element of each inner list is a subset of $\{true, false, FAIL\}$. If the subset is $\{true\}$ then the equilibrium (the first element of that list) is GAS. If the subset contains $false$ then the K value that was input does not work (and it is proven to not work using **LCoeff** or **Const**). If the subset contains $FAIL$ (but not $false$) then the K value is not proven to work, but it is also not proven to not work. Increasing N may help.

Try:

```
ProveK((4+x[n])/(1+x[n-1]), [x[n-1], x[n]], 5, 2);
```

Prove: A procedure to prove that a K value between **MinK** and **MaxK** works for a specific rational difference equation.

Inputs: R, vars, MinK, MaxK, N

Output: Set of lists of length 2. The first element of each list is the equilibrium, the second element of each list is the K value. If $MinK \leq K \leq MaxK$ then the equilibrium is GAS, and K is proven to work. If $K = 0$ then $MaxK$ is not large enough. If $K = -1$ then the equilibrium is not LAS.

Try:

`Prove((4+x[n])/(1+x[n-1]), [x[n-1], x[n]], 1, 5, 2);`

WebBook: A procedure to find K values for `numParams` sets of parameter values for a rational difference with parameters for coefficients, e.g.,

$$x_{n+1} = \frac{x_n}{A + x_{n-1}}$$

Inputs: `R`, `vars`, `params`, `paramPoss`, `numParams`, `MinK`, `MaxK`, `N`

`R` - a rational difference equation (with parameters)

`params` - parameters in `R`

`paramPoss` - possible values for parameters

`numParams` - number of parameter sets to investigate

Outputs: Set of lists of length 2. The first element is the set of parameters that were tested, the second element is the output of the procedure `Prove` for `R` with the given parameters.

Try:

`WebBook(x[n]/(A+x[n-1]), [x[n-1], x[n]], [A],
{seq(i/10, i=1..50)}, 3, 1, 5, 2);`

2.7 Conclusion

In Sections 2.2 and 2.3 we have seen both parts of my GAS algorithm: first reducing the problem to proving that a polynomial is positive, and then proving polynomial positivity. Putting the two together we now have a completely algorithmic approach to proving GAS of a given rational difference equation.

Inputs:

R - rational function in $k + 1$ variables

\bar{x} - equilibrium, solution to $\bar{x} = R(\bar{x}, \dots, \bar{x})$

$MaxK$ - a maximum K value to try

Outputs:

true if \bar{x} is proven to be GAS for $x_{n+1} = R(x_n, x_{n-1}, \dots, x_{n-k})$

false if \bar{x} is not LAS for $x_{n+1} = R(x_n, x_{n-1}, \dots, x_{n-k})$

FAIL if $MaxK$ was not high enough.

Procedure:

1. Check local asymptotic stability using Theorem 2.1.1 (Linearized Stability Theorem). If not LAS then output **false**. If LAS then continue to Step 2.
2. Conjecture a K value that satisfies Theorem 2.1.10 using the procedure outlined in Section 2.2.
3. Apply the n -dimensional polynomial positivity algorithm outlined at the end of Section 2.3. If the conjectured K value was proven to work, output **true**. If the conjectured K value was proven *not* to work (P_K failed **LCoeff** or **Const**), or the algorithm reached a recursion limit, continue to Step 4.
4. If $K < MaxK$, increment K by 1 and return to Step 3. If $K \geq MaxK$ then output **FAIL**.

This algorithm now gives a completely automatic proof machine for global asymptotic stability. As was mentioned in Section 2.1.2, this problem has historically not been approached in any kind of systematic fashion. Many of the theorems stated in Section 2.1.2, along with countless others appearing in [3, 24], were developed as generalizations of techniques used to prove GAS of specific difference equations. This meant that given a particular difference equation, proving its equilibrium is GAS would amount to trying to apply various known theorems; there may not have been a clear cut path leading to the proof. My algorithm can now serve as that path. Of course, given a difference equation that is known to be GAS, my algorithm may not always be able to prove it. However, I believe that it is much more widely applicable than any one previously known theorem guaranteeing global asymptotic stability.

The results that my algorithm can prove, for order 2 rational difference equations with linear numerator and denominator and parameter coefficients, were shown in Section 2.5. They are also summarized in a table in Appendix A.1. In addition, I have run

my Maple procedure `WebBook` (described in Section 2.6) which has generated hundreds of GAS theorems when specific parameter values are given. Some of these are summarized in Appendix A.2, and many more can be found on my website in `WebBooks`.

Chapter 3

A New Family of Somos-Like Recurrences

In the previous chapter we developed an algorithm to prove the convergence of sequences produced by rational difference equations (also known as rational *recurrences*). The algorithm was developed, and subsequently programmed, in order to systematically generate *proofs*. In this chapter, though we do mention the use of computers to come up with conjectures (see Section 3.2.1) the role of computers is less essential.

Additionally, in the previous chapter we were concerned with studying the *end behavior* of a sequence produced by a rational recurrence. Here, and in Chapter 4, the topic of interest will be the surprising nature of the terms themselves. Generally, when one considers a rational recurrence with integer initial conditions, the sequence will consist of rational numbers. But as we will see, occasionally we observe integer sequences being produced. There can be many explanations for this phenomenon, some of which will be described in this chapter.

3.1 Introduction to Somos-type Recurrences

3.1.1 Somos Sequences

The study of integer sequences produced by nonlinear recurrences took off when Michael Somos introduced the following recurrence

$$s_n s_{n-6} = s_{n-1} s_{n-5} + s_{n-2} s_{n-4} + s_{n-3}^2, \quad \text{with } s_i = 1 \text{ for } 1 \leq i \leq 6 \quad (3.1)$$

in Problem 1470 in the 1989 volume of the *Crux Mathematicorum* [31]. This recurrence produces what is now sequence A006722 in the OEIS [29]:

$$1, 1, 1, 1, 1, 1, 3, 5, 9, 23, 75, 421, 1103, 5047, 41783, 281527, 2534423, \dots$$

The problem Somos proposed was to prove (or disprove) that $s_n \in \mathbb{Z}$ for all $n \geq 1$. Somos discovered this recurrence through his study of elliptic theta functions [32]. When Somos first discovered this recurrence he verified that $s_n \in \mathbb{Z}$ for the first hundred terms. This was quite surprising since in order to compute s_n you must divide a combination of previous terms by s_{n-6} . One would expect a recurrence of this form to produce a sequence of rational numbers, not integers. As we will see, this recurrence led to the study of many related recurrences that share this integrality property. In a 1991 expository article [14], David Gale surveys the research inspired by this recurrence for the first few years after it was introduced. Much of what follows in this introduction can be found in his article.

No immediate proofs were given for integrality of Somos' original recurrence (3.1), so the recurrence was generalized in hopes that this would lead to a proof. This generalization, known as the Somos- k sequence, is given by the following recurrence:

$$s_n s_{n-k} = \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} s_{n-i} s_{n-k+i},$$

with initial conditions $s_m = 1$ for $1 \leq m \leq k$. Notice that $k = 6$ yields (3.1). The integrality phenomenon was also observed when $k = 4, 5$, and 7 (and it is easy to show that $k = 2, 3$ yields $s_n = 1$ for all n). However, for $k \geq 8$ we quickly see non-integer rational numbers in the sequence (see A030127 in [29]). Soon after this generalization, a number theoretic proof of integrality for Somos-4 was given by Janice Malouf [27]. Here I will present the variation due to George Bergman that can be found in the Gale article [14].

Proposition 3.1.1. *Let the sequence $\{a_n\}_{n=1}^{\infty}$ be produced by the following recurrence:*

$$a_n a_{n-4} = a_{n-1} a_{n-3} + a_{n-2}^2, \tag{3.2}$$

with initial conditions $a_i = 1$ for $1 \leq i \leq 4$. Then $a_n \in \mathbb{Z}$ for all $n \geq 1$.

Proof. First, we show that any four consecutive terms are pairwise relatively prime. If not, let n be the smallest index for which a_n has a common factor with one of a_{n-1} , a_{n-2} , or a_{n-3} . Assume that a_n and a_{n-1} share a common prime factor, p . Because

p divides $a_n a_{n-4}$ and $a_{n-1} a_{n-3}$, p must also divide a_{n-2}^2 (and hence, a_{n-2}). But this is a contradiction to the minimality of n . In the case that a_n and a_{n-2} share a common prime factor, p , we see analogously that a contradiction arises when p must divide a_{n-1} or a_{n-3} .

We will now show by induction on n that if $a_{n-4}, \dots, a_n, \dots, a_{n+3}$ are integers, then $a_{n+4} \in \mathbb{Z}$. The base case, for $n = 5$, is clearly true because our initial conditions are all 1. Assume, as the inductive hypothesis, that $a_{n-4}, \dots, a_{n+3} \in \mathbb{Z}$ for some $n \geq 5$. For clarity in notation, let $a_{n-3} := a$, $a_{n-2} := b$, and $a_{n-1} := c$. Then, since a_{n-4} is assumed to be an integer we must have that a_n divides $a_n a_{n-4} = ac + b^2$, so $ac + b^2 \equiv 0 \pmod{a_n}$. Because a , b , c , and a_n are pairwise relatively prime we can apply the recurrence (3.2) modulo a_n using modular division:

$$\begin{aligned} a_{n+1} &= \frac{a_n a_{n-2} + a_{n-1}^2}{a_{n-3}} \equiv \frac{0 \cdot b + c^2}{a} \equiv \frac{c^2}{a} \pmod{a_n} \\ a_{n+2} &= \frac{a_{n+1} a_{n-1} + a_n^2}{a_{n-2}} \equiv \frac{\frac{c^2}{a} c + 0^2}{b} \equiv \frac{c^3}{ab} \pmod{a_n} \\ a_{n+3} &= \frac{a_{n+2} a_n + a_{n+1}^2}{a_{n-1}} \equiv \frac{\frac{c^3}{ab} \cdot 0 + \left(\frac{c^2}{a}\right)^2}{c} \equiv \frac{c^3}{a^2} \pmod{a_n}. \end{aligned}$$

Finally, we have

$$\begin{aligned} a_{n+4} a_n &= a_{n+3} a_{n+1} + a_{n+2}^2 \in \mathbb{Z} \quad (\text{by inductive hypothesis}) \\ &\equiv \frac{c^3}{a^2} \frac{c^2}{a} + \left(\frac{c^3}{ab}\right)^2 \\ &\equiv \frac{c^5 b^2}{a^3 b^2} + \frac{c^6 a}{a^3 b^2} \equiv \frac{c^5}{a^3 b^2} (ac + b^2) \equiv 0 \pmod{a_n}. \end{aligned}$$

We see that $a_{n+3} a_{n+1} + a_{n+2}^2$ is divisible by a_n , and so a_{n+4} is an integer. Therefore, by induction, $a_n \in \mathbb{Z}$ for all n . \square

This proof of integrality of the Somos-4 sequence immediately generalizes to show that the Somos-5 recurrence produces an integer sequence. However, for $k = 6$ and 7 the proof fails in the beginning. We cannot establish an equivalent pairwise relatively prime statement since there are more than three terms in the Somos-6 and 7 recurrences. Even though this proof does not generalize, proofs for $k = 6$ and 7 are known using the Laurent phenomenon (see Section 3.1.2) and enumeration (Section 3.1.3).

3.1.2 Laurent Phenomenon

The first proof of integrality of the Somos-6 sequence given by recurrence (3.1) was found by Dean Hickerson using the computer algebra system Macsyma. He established the stronger result that if $s_i = x_i$, a formal variable for $1 \leq i \leq 6$, then s_n is a Laurent polynomial: a rational function in which the denominator is a monomial in the initial 6 variables, i.e., $s_n \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_6^{\pm 1}]$. This, of course, proves integrality by letting $x_i = 1$ for $1 \leq i \leq 6$. The Laurent property is shared by all Somos- k recurrences that produce integers. In general, when a recurrence produces a sequence of Laurent polynomials in its initial conditions, we say that it possesses the *Laurent phenomenon* [11]. A broad understanding of this phenomenon was first developed by Fomin and Zelevinsky through their study of cluster algebras (see [1, 9, 10, 12, 13]).

Eventually, we will see Fomin and Zelevinsky's theorem which gives sufficient conditions for a recurrence to produce a sequence of Laurent polynomials. These conditions, on the surface, seem quite arbitrary. Therefore I will first give some motivation to explain the role cluster algebras play in proving the Laurent property. For a more in-depth motivation see [11, 35]. In the most general setting, a cluster algebra can be thought of as collections of variables (called *clusters*) associated with vertices in a tree, T , along with polynomials, $P_1, \dots, P_k \in \mathbb{Z}[y_1, \dots, y_k]$ with P_i not depending on y_i , associated to edges of the tree (called *exchange polynomials*). For each vertex $t \in T$ we associate the cluster $\mathbf{x}(t) = \{x_1(t), \dots, x_k(t)\}$. We choose one vertex $t_0 \in T$ to be the root and call its cluster the *initial cluster*, $\mathbf{x}(t_0)$. The polynomials associated to the edges of T give us a way to express the variables in any cluster in terms of variables in the initial cluster. If (t, t') is an edge in T with associated polynomial P_i then we have the following *exchange relation* between clusters $\mathbf{x}(t)$ and $\mathbf{x}(t')$:

$$\begin{aligned} x_j(t) &= x_j(t') \text{ for any } j \neq i, \\ x_i(t) &= \frac{P_i(\mathbf{x}(t))}{x_i(t')}. \end{aligned}$$

Notice that this exchange relation strongly resembles the form of the Somos- k recurrence.

In order to see the connection to sequences and the Laurent phenomenon, and to

motivate Theorem 3.1.2 found below, we must consider the caterpillar tree $T_{k,l}$ which has l vertices of degree k on its spine. The tree $T_{3,4}$ is pictured in Figure 3.1. At each

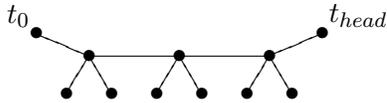


Figure 3.1: The caterpillar tree $T_{3,4}$

spine vertex the exchange polynomials, $P_1, \dots, P_k \in \mathbb{Z}[y_1, \dots, y_k]$, on the k incident edges must be distinct.

In [11], Fomin and Zelevinsky prove the Caterpillar Lemma (a generalization of a theorem found in [10] of the same name). The Caterpillar Lemma gives conditions for the exchange polynomials on $T_{k,l}$ which, if satisfied, guarantee that each cluster variable $x_i(t)$, for $i \in [k]$ and $t \in T_{k,l}$, is a Laurent polynomial in the variables $x_1(t_0), \dots, x_k(t_0)$, with coefficients in \mathbb{Z} . What will be important to us is the fact that the conditions in the Caterpillar Lemma can be satisfied when the exchange polynomials are created, in some canonical fashion, from a recurrence of the form

$$r_{n+k}r_n = F(r_{n+1}, \dots, r_{n+k-1}),$$

with initial conditions $r_i = x_i(t_0)$ for $1 \leq i \leq k$. In this case, one can show that the cluster variables associated to the vertex t_{head} , after being expressed in terms of the initial cluster, will be k consecutive terms in the sequence $\{r_n\}_{n=1}^{\infty}$. Of course, it is not true in general that the Caterpillar Lemma is satisfied for *any* recurrence of the form $r_{n+k}r_n = F(r_{n+1}, \dots, r_{n+k-1})$, but in many of the cases that we care about (e.g., Somos- k) the conditions are satisfied. With this cluster algebra setting in mind as motivation, we will now build up the machinery from [11] to see sufficient conditions on the recurrence that guarantee this Laurent property. In the end we will have defined a sequence of polynomials, $G_{k-1}, G_{k-2}, \dots, G_0$, recursively. One of the sufficient conditions will pertain to the polynomial G_0 . We will then use these sufficient conditions in Section 3.3 to prove that a large family of recurrences possesses the Laurent phenomenon.

First, we consider a recurrence in the following form

$$x_{n+k}x_n = F(x_{n+1}, \dots, x_{n+k-1}),$$

where $F \in \mathbb{Z}[y_1, \dots, y_{k-1}]$. In the case of Somos-4, $k = 4$, and $F(y_1, y_2, y_3) = y_1y_3 + y_2^2$. Note that this is not the most general way to define a nonlinear (or even rational) recurrence, but it is the required form to be able to apply Fomin and Zelevinsky's method. Next, define the polynomials $F_1, \dots, F_k \in \mathbb{Z}[y_1, \dots, y_k]$ as

$$F_m = F(y_{m+1}, \dots, y_{k-1}, y_k, y_1, \dots, y_{m-1}).$$

Since $m \in \{1, \dots, k\}$ this polynomial is well defined. For example, if $k = 4$ we have the four polynomials:

$$\begin{aligned} F_1 &= F(y_2, y_3, y_4), & F_2 &= F(y_3, y_4, y_1), \\ F_3 &= F(y_4, y_1, y_2), & F_4 &= F(y_1, y_2, y_3). \end{aligned}$$

Notice that F_m does not depend on y_m . These F_m play the role of the exchange polynomials on the spine of $T_{k,l}$. For $m \in [k-1]$ define the polynomial Q_m as

$$Q_m = F_m|_{y_k=0} = F(y_{m+1}, \dots, y_{k-1}, 0, y_1, \dots, y_{m-1}).$$

We are now ready to recursively define the polynomials $G_{k-1}, G_{k-2}, \dots, G_0$, which serve as exchange polynomials on the "legs" of the Caterpillar tree $T_{k,l}$. The polynomial G_{m-1} will be defined from G_m , using intermediate polynomials \tilde{G}_{m-1} and $\tilde{\tilde{G}}_{m-1}$. First, let $G_{k-1} := F$. Then assume we have defined G_m for some $1 \leq m \leq k-1$. From G_m first define \tilde{G}_{m-1} :

$$\tilde{G}_{m-1} := G_m|_{y_m = \frac{Q_m}{y_m}}.$$

Now, let L be a Laurent monomial in y_1, \dots, y_{k-1} such that $\frac{\tilde{G}_{m-1}}{L}$ is a polynomial in $\mathbb{Z}[y_1, \dots, y_{k-1}]$ not divisible by any non-unit in \mathbb{Z} . Then

$$\tilde{\tilde{G}}_{m-1} := \frac{\tilde{G}_{m-1}}{L}.$$

Next, let $b \in \mathbb{Z}$ be the maximal power such that Q_m^b divides $\tilde{\tilde{G}}_{m-1}$. Finally, we define G_{m-1} as

$$G_{m-1} := \frac{\tilde{\tilde{G}}_{m-1}}{Q_m^b}.$$

Now, we are able to restate Theorem 3.1 from [11]. The conditions in this theorem, when thought of in the setting of the Caterpillar tree, can be shown to be equivalent to the conditions in the Caterpillar Lemma found in [11].

Theorem 3.1.2 (Fomin, Zelevinsky 2002). *Let $F \in \mathbb{Z}[y_1, \dots, y_{k-1}]$ be a polynomial satisfying the following conditions:*

- (i) F is not divisible by any y_i for $i \in [k-1]$,
- (ii) each Q_m is an irreducible element of $\mathbb{Z}[y_1^{\pm 1}, \dots, y_{k-1}^{\pm 1}]$, and
- (iii) $G_0 = F$.

Then every term of the sequence $\{x_n\}_{n=1}^{\infty}$, defined by the recurrence

$$x_{n+k} = \frac{F(x_{n+1}, \dots, x_{n+k-1})}{x_n},$$

is a Laurent polynomial in the initial k terms, with coefficients in \mathbb{Z} .

Let us see how one can use this theorem to prove Laurentness of the Somos-4 sequence. As stated before, the Somos-4 recurrence is given by $F(y_1, y_2, y_3) = y_1 y_3 + y_2^2$ and $k = 4$. First, we see that F is not divisible by any y_i for $i \in \{1, 2, 3\}$, so F satisfies condition (i). Next, we verify requirement (ii) by constructing Q_1, Q_2 , and Q_3 :

$$Q_1 = F_1|_{y_4=0} = F(y_2, y_3, 0) = y_2 \cdot 0 + y_3^2 = y_3^2,$$

$$Q_2 = F_2|_{y_4=0} = F(y_3, 0, y_1) = y_3 \cdot y_1 + 0^2 = y_3 y_1,$$

$$Q_3 = F_3|_{y_4=0} = F(0, y_1, y_2) = 0 \cdot y_2 + y_1^2 = y_1^2.$$

Since y_1, y_2, y_3 are unit elements in $\mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}]$, the monomials Q_i are irreducible.

Finally, we must show (iii) by constructing G_3, \dots, G_0 and verifying that $G_0 = F$. First we set $G_3 := F = y_1 y_3 + y_2^2$. The rest is straightforward, so just stating the polynomials will suffice.

$$\begin{array}{ccc}
n = 2 & n = 1 & n = 0 \\
\tilde{G}_2 := G_3|_{y_3 = \frac{Q_3}{y_3}} & \tilde{G}_1 := G_2|_{y_2 = \frac{Q_2}{y_2}} & \tilde{G}_0 := G_1|_{y_1 = \frac{Q_1}{y_1}} \\
= y_1^3 y_3^{-1} + y_2^2 & = y_1^3 + y_1^2 y_2^{-2} y_3^3 & = y_1^{-1} y_2^2 y_3^2 + y_3^3 \\
\tilde{\tilde{G}}_2 := \frac{\tilde{G}_2}{y_3^{-1}} = y_1^3 + y_2^2 y_3 & \tilde{\tilde{G}}_1 := \frac{\tilde{G}_1}{y_1^2 y_2^{-2}} = y_1 y_2^2 + y_3^3 & \tilde{\tilde{G}}_0 := \frac{\tilde{G}_0}{y_1^{-1} y_2^2} = y_2^2 + y_1 y_3 \\
G_2 := \frac{\tilde{\tilde{G}}_2}{Q_3^0} = y_1^3 + y_2^2 y_3 & G_1 := \frac{\tilde{\tilde{G}}_1}{Q_2^0} = y_1 y_2^2 + y_3^3 & G_0 := \frac{\tilde{\tilde{G}}_0}{Q_1^0} = y_1 y_3 + y_2^2 = F
\end{array}$$

Since $G_0 = F$ we know by Theorem 3.1.2 that the Somos-4 recurrence generates a sequence of Laurent polynomials.

3.1.3 Somos and Enumeration

One thing that the number theoretic and Laurent phenomenon proofs lack is a reason for integrality, other than some lucky cancellation. Another way to prove integrality of a sequence is to find some objects that the numerical sequence counts. If one can establish an enumerative proof, then we will have a much more tangible reason for integrality. This was done for the Somos-4 and 5 sequences by David Speyer [33], and for Somos-6 and 7 by Gabriel Carroll and Speyer [4]. The Somos-4 and 5 sequences count perfect matchings of a sequence of graphs that grow as n grows; see Figures 3.2 and 3.3 for the first few graphs (vertices of the graph are when two lines cross, or when there is a corner) [33]. For the Somos-6 and 7 sequences, Carroll and Speyer proved that the number of “groves” (collections of a particular type of restricted path in a triangular lattice) of size n corresponds to the n^{th} element in the sequence.

3.1.4 Somos-inspired Recurrences

Now that the integrality of the Somos- k sequences has been established for $k = 4, 5, 6, 7$, and they are known to be non-integral for $k \geq 8$, it might seem that this topic is completely understood. However, these recurrences inspired numerous related recurrences with the same integrality and Laurent properties, many of which are discussed in [14].

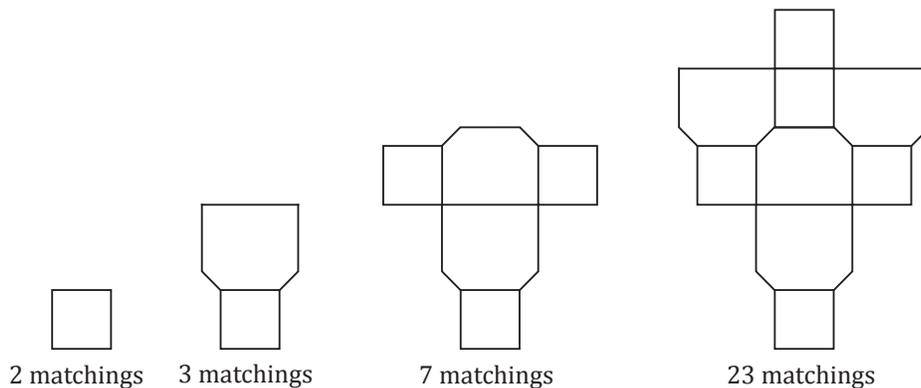


Figure 3.2: The first four nontrivial Somos-4 graphs

Gale and Raphael Robinson generalized Somos-4 and 5 with the following recurrence:

$$a_n a_{n-k} = x a_{n-l} a_{n-k+l} + y a_{n-m} a_{n-k+m}, \quad (3.3)$$

where $x, y \in \mathbb{Z}$ and $1 < l < m < k$. We obtain Somos-4 when we let $k = 4$, $l = 1$, $m = 2$, and $x = y = 1$. Similarly, by letting $k = 5$, $l = 1$, $m = 2$, and $x = y = 1$, we get Somos-5. Recurrence (3.3) is known as the *three-term Gale-Robinson recurrence*. We can similarly generalize Somos-6, 7 with the *four-term Gale-Robinson recurrence*:

$$a_n a_{n-k} = x a_{n-p} a_{n-k+p} + y a_{n-q} a_{n-k+q} + z a_{n-r} a_{n-k+r}, \quad (3.4)$$

where $x, y, z \in \mathbb{Z}$ and $p+q+r = k$. Integrality and Laurentness are known for both (3.3) and (3.4) for all $x, y, z \in \mathbb{Z}$. In [4, 33], Carroll and Speyer actually gave enumerative proofs for these recurrences. The combinatorial objects mentioned in the previous section are merely byproducts of this more general result.

Up to this point, all of the recurrences we have seen in this chapter are homogeneous, in the sense that all of the terms have the same total degree. However, this doesn't have to be the case. As Gale recounts in his story of the Somos sequences, Dana Scott wrote a program to study (3.2) but forgot to square the a_{n-2} term, yielding the recurrence

$$a_n a_{n-4} = a_{n-1} a_{n-3} + a_{n-2}. \quad (3.5)$$

Much to his surprise, (3.5) still produced a sequence of integers when the initial conditions were $a_i = 1$ for $1 \leq i \leq 4$. This curiosity led to the generalization

$$a_n a_{n-4} = a_{n-1}^p a_{n-3}^q + a_{n-2}^r,$$

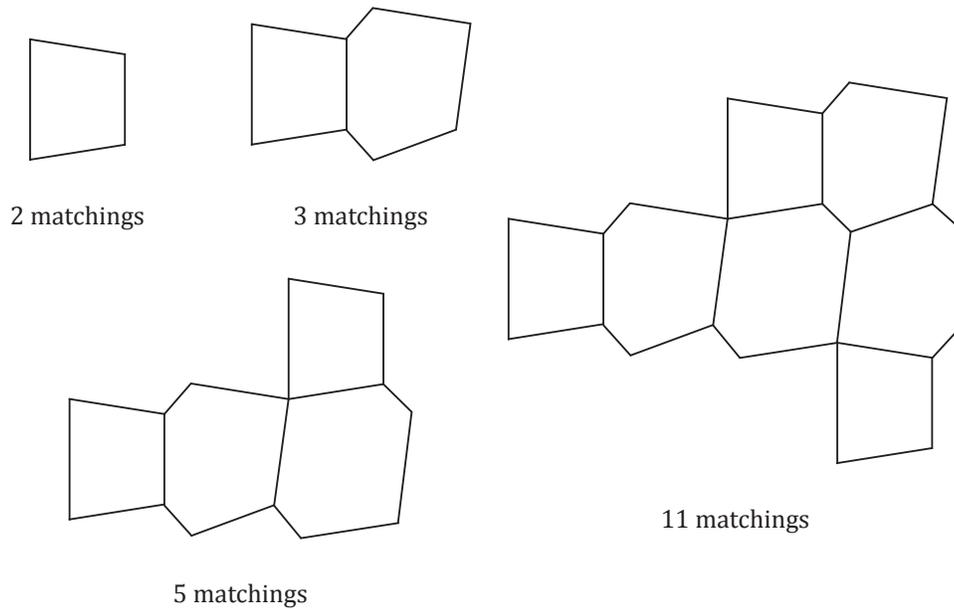


Figure 3.3: The first four nontrivial Somos-5 graphs

which yields integers for any $p, q, r \in \mathbb{N}$. Paul Heideman and I took the idea of (3.5) in a different direction. We considered the following non-homogeneous recurrence:

$$h_n h_{n-(2K+1)} = h_{n-1} h_{n-2K} + h_{n-K} + h_{n-(K+1)}, \quad (3.6)$$

with initial conditions $h_i = 1$ for $1 \leq i \leq k$. In our paper, Heideman and I prove that for any fixed $K \in \mathbb{N}$, the sequence produced by (3.6) consists of integers [17]. To do this, we show that it is also produced by the linear recurrence

$$h_n = [2K^2 + 8K + 4] (h_{n-2K} - h_{n-4K}) + h_{n-6K}, \quad (3.7)$$

with the initial $6K + 1$ terms given by the quadratic recurrence (3.6). Of course, to prove integrality of the sequence, the initial $6K + 1$ terms must be integers. If not, the linear recurrence (3.7) does not produce an integer sequence. Therefore, we also show that these initial terms are given by a piecewise polynomial with integer coefficients, and are therefore integers. The following two lemmas and proofs appeared in [17]. We restate the proofs here because similar lemmas appear in Section 3.2. The proofs of the lemmas in 3.2 are nearly identical to those presented here and so they will be omitted from Section 3.2.

Lemma 3.1.3. *The initial $6K + 1$ terms of the sequence produced by (3.6) are given by the following piecewise polynomial:*

$$\begin{aligned}
h_m &= 1 & 1 \leq m \leq 2K + 1 \\
h_{2K+m} &= 2m - 1 & 2 \leq m \leq K + 1 \\
h_{3K+m} &= 2m^2 + 2K - 2m + 1 & 2 \leq m \leq K + 1 \\
h_{4K+m} &= 4K^2m - 2K^2 + 12Km + 2m^2 - 8K + 2m - 3 & 2 \leq m \leq K + 1 \\
h_{5K+m} &= 4K^2m^2 + 4K^3 - 4K^2m + 16Km^2 + \\
&\quad + 16K^2 - 16Km + 8m^2 + 10K - 10m + 3 & 2 \leq m \leq K + 1.
\end{aligned}$$

Proof. The first of the piecewise polynomials, $h_m = 1$ for $1 \leq m \leq 2K + 1$, is simply the definition of the initial conditions for (3.6). Each of the other relations is proved independently by induction. These polynomials were originally conjectured and proved via a computer program written by Doron Zeilberger.

We will now show that $h_{2K+m} = 2m - 1$ for $2 \leq m \leq K + 1$. For the base case, we must verify that $h_{2K+2} = 3$. From (3.6) we have

$$\begin{aligned}
h_{2K+2}h_1 &= h_{2K+1}h_2 + h_{K+2} + h_{K+1} \\
h_{2K+2} &= 1 \cdot 1 + 1 + 1 = 3.
\end{aligned}$$

Now we assume that $h_{2K+m} = 2m - 1$ for some $2 \leq m < K + 1$. We need to show that $h_{2K+(m+1)} = 2(m+1) - 1 = 2m + 1$ for $2 \leq m \leq K$. We will use the definition of $h_{2K+(m+1)}$ from (3.6):

$$h_{2K+(m+1)}h_m = h_{2K+m}h_{m+1} + h_{K+m+1} + h_{K+m}.$$

Since $2 \leq m \leq K$, we know that

$$\begin{aligned}
3 &\leq m + 1 \leq K + 1 \\
K + 3 &\leq K + (m + 1) \leq 2K + 1 \\
K + 2 &\leq K + m \leq 2K.
\end{aligned}$$

From the definition of h_i for $1 \leq i \leq 2K + 1$ we have that $h_{m+1} = h_{K+(m+1)} = h_{K+i} = 1$. From the inductive hypothesis, we also know that $h_{2K+m} = 2m - 1$. Plugging these

values in, we have

$$h_{2K+(m+1)} \cdot 1 = (2m - 1) \cdot 1 + 1 + 1$$

$$h_{2K+(m+1)} = 2m + 1.$$

So, by induction, $h_{2K+m} = 2m - 1$ for $2 \leq m \leq K + 1$. Notice that this is induction on a finite set; the induction variable m is restricted between 2 and $K + 1$, and the induction hypothesis is not true when $m \geq K + 1$.

The rest of the piecewise polynomial proofs follow the same general form, and are thus omitted. \square

Lemma 3.1.4. *The sequence produced by (3.6) also satisfies the linear recurrence (3.7) for $n \geq 6K + 2$.*

Proof. First note that proving the converse (i.e., that the sequence given by the linear recurrence (3.7) satisfies the quadratic recurrence (3.6)) is equivalent to proving the statement itself. To see this in general, let $\{x_n\}_{n=1}^{\infty}$ be produced by the recurrence $x_n = F_1(x_{n-1}, \dots, x_{n-k_1})$ with initial conditions x_1, \dots, x_{k_1+1} . If this sequence also satisfies $x_n = F_2(x_{n-1}, \dots, x_{n-k_2})$ then the sequence produced by F_2 , with initial conditions given by F_1 , satisfies F_1 simply by uniqueness of the sequence. Thus we will assume that $\{h_n\}$ is given by (3.7), and show that it is annihilated by (3.6) using strong induction.

Define the sequence $\{h_n\}$ for all $n \geq 6K + 2$ by (3.7), and let h_n for $1 \leq n \leq 6K + 1$ be given by the piecewise polynomial in Lemma 3.1.3. To show that (3.6) annihilates the sequence produced by (3.7), let

$$\phi(n) = h_n h_{n-(2K+1)} - h_{n-1} h_{n-2K} - h_{n-K} - h_{n-K-1}. \quad (3.8)$$

We will prove by induction that $\phi(n) = 0$, for all $n \geq 2K + 2$. Clearly $\phi(n) = 0$ for $2K + 2 \leq n \leq 6K + 1$ since the first $6K + 1$ terms coming from Lemma 3.1.3 have been shown to be produced by (3.6).

For the base case, we must prove that $\phi(6K + 2) = 0$. This is nothing but algebraic calculations, easily verified by a computer algebra system such as Maple or Mathematica. For all terms except h_{6K+2} we can substitute the initial conditions for the linear

recurrence into the definition of $\phi(6K + 2)$:

$$\phi(6K + 2) = h_{6K+2}h_{4K+1} - h_{6K+1}h_{4K+2} - h_{5K+2} - h_{5K+1}.$$

These initial conditions are given by

$$h_{4K+1} = h_{3K+(K+1)} = 2K^2 + 4K + 1$$

$$h_{6K+1} = h_{5K+(K+1)} = 4K^4 + 24K^3 + 40K^2 + 16K + 1$$

$$h_{4K+2} = 6K^2 + 16K + 9$$

$$h_{5K+2} = 4K^3 + 24K^2 + 42K + 15$$

$$h_{5K+1} = h_{4K+(K+1)} = 4K^3 + 6K^2 + 10K + 1.$$

However, h_{6K+2} must be computed from the linear recurrence (3.7):

$$\begin{aligned} h_{6K+2} &= (2K^2 + 8K + 4)(h_{4K+2} - h_{2K+2}) + h_2 \\ &= (2K^2 + 8K + 4)((6K^2 + 16K + 9) - 3) + 1 \\ &= 12K^4 + 80K^3 + 164K^2 + 112K + 25. \end{aligned}$$

Now it is a matter of plugging this into a computer algebra system and verifying that $\phi(6K + 2) = 0$.

Since the base case is verified, we can proceed with the induction. We make the strong induction assumption that $\phi(m) = 0$ for all $m < n$, and we need to show that $\phi(n) = 0$. We compute $\phi(n)$ by substituting for h_n , h_{n-1} , h_{n-K} , and h_{n-K-1} from the definition of $\{h_n\}$:

$$h_n = H(K)h_{n-2K} - H(K)h_{n-4K} + h_{n-6K}$$

$$h_{n-1} = H(K)h_{n-2K-1} - H(K)h_{n-4K-1} + h_{n-6K-1}$$

$$h_{n-K} = H(K)h_{n-3K} - H(K)h_{n-5K} + h_{n-7K}$$

$$h_{n-K-1} = H(K)h_{n-3K-1} - H(K)h_{n-5K-1} + h_{n-7K-1}.$$

For simplicity in notation, we have let the term $2K^2 + 8K + 4$ be denoted by $H(K)$.

After substituting into $\phi(n)$, expand and then simplify and we are left with

$$\begin{aligned}\phi(n) = & -H(K)h_{n-4K}h_{n-2K-1} + H(K)h_{n-4K-1}h_{n-2K} - H(K)h_{n-3K} + \\ & -H(K)h_{n-3K-1} + h_{n-2K-1}h_{n-6K} - h_{n-2K}h_{n-6K-1} + H(K)h_{n-5K} + \\ & + H(K)h_{n-5K-1} - h_{n-7K} - h_{n-7K-1}.\end{aligned}$$

If we collect terms (with the intention to apply the inductive hypothesis) we see that

$$\begin{aligned}-H(K)h_{n-4K}h_{n-2K-1} + H(K)h_{n-4K-1}h_{n-2K} - H(K)h_{n-3K} + \\ -H(K)h_{n-3K-1} = -H(K)\phi(n-2K)\end{aligned}$$

which equals 0 by the induction hypothesis. Thus

$$\begin{aligned}\phi(n) = & h_{n-2K-1}h_{n-6K} - h_{n-2K}h_{n-6K-1} - h_{n-7K} - h_{n-7K-1} \\ & + H(K)h_{n-5K} + H(K)h_{n-5K-1}.\end{aligned}$$

Now we substitute h_{n-2K} and h_{n-2K-1} from the definition of $\{h_n\}$:

$$\begin{aligned}h_{n-2K} = & H(K)h_{n-4K} - H(K)h_{n-6K} + h_{n-8K} \\ h_{n-2K-1} = & H(K)h_{n-4K-1} - H(K)h_{n-6K-1} + h_{n-8K-1}.\end{aligned}$$

Simplifying again, we obtain:

$$\begin{aligned}\phi(n) = & H(K)h_{n-4K-1}h_{n-6K} - H(K)h_{n-4K}h_{n-6K-1} - H(K)h_{n-5K} - H(K)h_{n-5K-1} + \\ & + h_{n-6K}h_{n-8K-1} - h_{n-6K-1}h_{n-8K} - h_{n-7K} - h_{n-7K-1} \\ = & -H(K)\phi(n-4K) + \phi(n-6K) \\ = & 0.\end{aligned}$$

Thus by induction, $\phi(n) = 0$ for all $n \in \mathbb{Z}^+$. □

Putting Lemmas 3.1.3 and 3.1.4 together, we obtain the following theorem.

Theorem 3.1.5. *The recurrence*

$$h_n h_{n-(2K+1)} = h_{n-1} h_{n-2K} + h_{n-K} + h_{n-(K+1)}, \quad (3.9)$$

with initial conditions $h_n = 1$ for $n \leq 2K+1$, generates an infinite sequence of integers.

After Heideman and I proved integrality of this particular sequence, we generalized in the same manner that Gale and Robinson generalized the Somos recurrence. Consider the following 3-parameter family of recurrences:

$$x_n x_{n-k} = x_{n-i} x_{n-k+i} + x_{n-j} + x_{n-k+j}, \quad (3.10)$$

where $i < k - i < k$, $j < k - j < k$, and initial conditions $x(l) = 1$ for $1 \leq l \leq k$. If we let k be odd ($k = 2K + 1$), $i = 1$, and $j = \frac{k-1}{2} = K$, we get (3.6). However, unlike Gale-Robinson, it is not true that (3.10) produces an integer sequence for any choice of k, i, j . Instead, we made the following conjecture.

Conjecture 3.1.6. *Consider the quadratic recurrence (3.10). This recurrence produces a sequence of integers iff one of the following holds:*

1. k is even, i is odd, and $j = \frac{k}{2}$,
2. k is even, i is even, and $j = \frac{i}{2}$, $j = \frac{k}{2}$, or $j = \frac{k-i}{2}$,
3. k is odd, i is odd, and $j = \frac{k-i}{2}$,
4. k is odd, i is even, and $j = \frac{i}{2}$.

In Section 3.2, we will see what progress has been made towards finding linear recurrences for the cases in Conjecture 3.1.6. Then, in Section 3.3 we will use the Laurent phenomenon to prove the backwards implication in this conjecture.

3.2 Finding Linear Annihilators for Quadratic Recurrences

In the previous section we saw that the recurrence (3.6) produces an integer sequence. After Heideman and I published [17], I began to look into proving more subcases of Conjecture 3.1.6. I was able to prove integrality of another special case of (3.10), analogous to (3.6), when $k = 2K$ is even. This recurrence is given by

$$e_n e_{n-2K} = e_{n-1} e_{n-(2K-1)} + e_{n-K} + e_{n-K}, \quad (3.11)$$

with initial conditions $e_n = 1$ for $1 \leq n \leq 2K$. We will prove integrality by first seeing that the initial $6K - 2$ terms of (3.11) are given by a piecewise polynomial. Then, we

see that the sequence produced by (3.11) is annihilated by a linear recurrence of order $6K - 3$. These two statements together prove integrality of the sequence $\{e_n\}_{n=1}^{\infty}$ in the same way that Lemmas 3.1.3 and 3.1.4 prove integrality of $\{h_n\}_{n=1}^{\infty}$.

Lemma 3.2.1. *The initial $6K - 2$ terms of the sequence produced by (3.11) are given by the following piecewise polynomial:*

$$\begin{aligned}
e_m &= 1 & 1 \leq m \leq 2K \\
e_{2K+m} &= 2m + 1 & 1 \leq m \leq K \\
e_{3K+m} &= 2m^2 + 2K + 4m + 1 & 1 \leq m \leq K - 1 \\
e_{4K+m} &= 4K^2m + 6K^2 + 8Km + 2m^2 + 10K + 2m - 1 & 0 \leq m \leq K - 1 \\
e_{5K+m} &= 4K^2m^2 + 4K^3 + 16K^2m + 12Km^2 + \\
&\quad + 24K^2 + 48Km - 2m^2 + 36K - 10m - 9 & 0 \leq m \leq K - 2
\end{aligned}$$

Lemma 3.2.2. *The sequence produced by (3.11) also satisfies the linear recurrence*

$$e_n = [2K^2 + 6K - 1] (e_{n-(2K-1)} - e_{n-(4K-2)}) + e_{n-(6K-3)}, \quad (3.12)$$

for $n \geq 6K - 2$.

The proofs are nearly identical to those of Lemmas 3.1.3 and 3.1.4, so they are omitted. Together these lemmas prove the following theorem.

Theorem 3.2.3. *The recurrence (3.11) generates an infinite sequence of integers.*

In (3.6) and (3.11) it is the case that $i = 1$ in (3.10). However, we can also prove integrality in some cases when i is arbitrary, and k, j are specific multiples of i . For example, if we let $k = (2K + 1)i$ and $j = Ki$ in (3.10) we get the recurrence

$$a_n a_{n-(2K+1)i} = a_{n-i} a_{n-2Ki} + a_{n-Ki} + a_{n-(K+1)i}. \quad (3.13)$$

We will now show that the sequence produced by (3.13) is simply an expansion of $\{h_n\}_{n=1}^{\infty}$ where each term is repeated exactly i times.

Theorem 3.2.4. *Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by the recurrence (3.13), and the sequence $\{h_n\}_{n=1}^{\infty}$ defined by the recurrence (3.6). Then for all integers $L \geq 0$,*

$$a_{Li+m} = h_{L+1}, \quad \text{for } m \in \{1, \dots, i\}. \quad (3.14)$$

Proof. We proceed by induction on L . For the base case, we must verify that (3.14) holds for $0 \leq L \leq 2K$. If L is in this range, $Li + m \in \{1, \dots, (2K + 1)i\}$, and $L + 1 \in \{1, \dots, 2K + 1\}$. Since this index puts us within the initial conditions for both recurrences,

$$a_{Li+m} = 1 = h_{L+1}.$$

Now, assume as the inductive hypothesis that $a_{Mi+m} = h_{M+1}$, for all $M < L$, and assume that $L > 2K$. Using the definition (3.13) of a_n and the inductive hypothesis we see

$$\begin{aligned} a_{Li+m}a_{Li+m-(2K+1)i} &= a_{Li+m-i}a_{Li+m-2Ki} + a_{Li+m-Ki} + a_{Li+m-(K+1)i} \\ &= a_{(L-1)i+m}a_{(L-2K)i+m} + a_{(L-K)i+m} + a_{(L-K-1)i+m} \\ &= h_{L-1+1}h_{L-2K+1} + h_{L-K+1} + h_{L-K-1+1} \\ a_{Li+m}h_{(L+1)-(2K+1)} &= h_{(L+1)-1}h_{(L+1)-2K} + h_{(L+1)-K} + h_{(L+1)-(K+1)}. \end{aligned}$$

Solving for a_{Li+m} and using the definition (3.6) of h_n we clearly see that

$$a_{Li+m} = \frac{h_{(L+1)-1}h_{(L+1)-2K} + h_{(L+1)-K} + h_{(L+1)-(K+1)}}{h_{(L+1)-(2K+1)}} = h_{L+1}.$$

Thus, by induction, $a_{Li+m} = h_{L+1}$. \square

In the proof of integrality for the sequence $\{h_n\}_{n=1}^{\infty}$ we showed that it is also annihilated by a linear recurrence of order $6K$. Since the sequence $\{a_n\}_{n=1}^{\infty}$ is $\{h_n\}_{n=1}^{\infty}$ with each term repeated i times, we also have a linear recurrence that annihilates $\{a_n\}_{n=1}^{\infty}$.

Proposition 3.2.5. *If the sequence $\{a_n\}_{n=1}^{\infty}$ is given by the recurrence (3.13) then it is also annihilated by the following linear recurrence of order $(6K - 1)i + 1$*

$$\begin{aligned} a_n &= a_{n-1} + \sum_{m_1=1}^{2K-1} (-a_{n-m_1i} + a_{n-(m_1i+1)}) + \\ &+ [2K^2 + 8K + 3] \sum_{m_2=2K}^{4K-1} (a_{n-m_2i} - a_{n-(m_2i+1)}) + \\ &+ \sum_{m_3=4K}^{6K-1} (-a_{n-m_3i} + a_{n-(m_3i+1)}). \end{aligned} \tag{3.15}$$

Proof. In contrast to the proof of Lemma 3.1.4, which establishes a linear recurrence for the sequence $\{h_n\}_{n=1}^{\infty}$, this proof will *not* be done using induction. Instead, we will use the linear recurrence for $\{h_n\}_{n=1}^{\infty}$ along with Theorem 3.2.4 to prove that $\{a_n\}_{n=1}^{\infty}$ is annihilated by (3.15). First, define $\phi(n)$ as follows:

$$\begin{aligned}\phi(n) := & a_n - a_{n-1} - \sum_{m_1=1}^{2K-1} (-a_{n-m_1 i} + a_{n-(m_1 i+1)}) + \\ & - [2K^2 + 8K + 3] \sum_{m_2=2K}^{4K-1} (a_{n-m_2 i} - a_{n-(m_2 i+1)}) + \\ & - \sum_{m_3=4K}^{6K-1} (-a_{n-m_3 i} + a_{n-(m_3 i+1)}).\end{aligned}$$

We will show that $\phi(n) = 0$ for all $n \geq (6K-1)i$ directly, using $a_{Li+m} = h_{L+1}$ and the fact that $\{h_n\}_{n=1}^{\infty}$ is annihilated by the linear recurrence (3.7).

In order to use Theorem 3.2.4 we must express n modulo i , i.e., $n = Li + m$ where $L \geq 0$ and $m \in \{1, \dots, i\}$. Then

$$\begin{aligned}\phi(Li + m) = & a_{Li+m} - a_{Li+(m-1)} - \sum_{m_1=1}^{2K-1} (-a_{(L-m_1)i+m} + a_{(L-m_1)i+(m-1)}) + \\ & - [2K^2 + 8K + 3] \sum_{m_2=2K}^{4K-1} (a_{(L-m_2)i+m} - a_{(L-m_2)i+(m-1)}) + \\ & - \sum_{m_3=4K}^{6K-1} (-a_{(L-m_3)i+m} + a_{(L-m_3)i+(m-1)}).\end{aligned}$$

In order to show that $\phi(Li + m) = 0$, we will need to treat the cases $m \in \{2, \dots, i\}$ and $m = 1$ separately. First we let $m \in \{2, \dots, i\}$, and use the fact that $a_{Li+m} = h_{L+1}$ from Theorem 3.2.4 to simplify $\phi(Li + m)$ in terms of h :

$$\begin{aligned}\phi(Li + m) = & h_{L+1} - h_{L+1} - \sum_{m_1=1}^{2K-1} (-h_{L-m_1+1} + h_{L-m_1+1}) + \\ & - [2K^2 + 8K + 3] \sum_{m_2=2K}^{4K-1} (h_{L-m_2+1} - h_{L-m_2+1}) + \\ & - \sum_{m_3=4K}^{6K-1} (-h_{L-m_3+1} + h_{L-m_3+1}) = 0.\end{aligned}$$

Now we must prove that $\phi(Li + 1) = 0$. As in the previous case we must first rewrite $\phi(Li + 1)$ so that all indices on a are of the form $Li + \tilde{m}$ with $\tilde{m} \in \{1, \dots, i\}$. Then

we simplify as before in terms of $\{h_n\}$. This case is slightly more complicated, since $a_{(L-m_k)i+m}$ does not equal $a_{(L-m_k)i+(m-1)}$ (as in the previous case). We first rewrite $\phi(Li+1)$, but in this case we sometimes have to rewrite $(L-m_j)i$ as $(L-m_j-1)i+i$:

$$\begin{aligned}\phi(Li+1) &= a_{Li+1} - a_{(L-1)i+i} - \sum_{m_1=1}^{2K-1} (-a_{(L-m_1)i+1} + a_{(L-m_1-1)i+i}) + \\ &\quad - [2K^2 + 8K + 3] \sum_{m_2=2K}^{4K-1} (a_{(L-m_2)i+1} - a_{(L-m_2-1)i+i}) + \\ &\quad - \sum_{m_3=4K}^{6K-1} (-a_{(L-m_3)i+1} + a_{(L-m_3-1)i+i}).\end{aligned}$$

Next, we use Theorem 3.2.4 to rewrite in terms of the h sequence:

$$\begin{aligned}\phi(Li+1) &= h_{L+1} - h_L - \sum_{m_1=1}^{2K-1} (-h_{L-m_1+1} + h_{L-m_1}) + \\ &\quad - [2K^2 + 8K + 3] \sum_{m_2=2K}^{4K-1} (h_{L-m_2+1} - h_{L-m_2}) + \\ &\quad - \sum_{m_3=4K}^{6K-1} (-h_{L-m_3+1} + h_{L-m_3}).\end{aligned}$$

Each of these sums are telescoping, so we have that

$$\begin{aligned}\sum_{m_1=1}^{2K-1} (-h_{L-m_1+1} + h_{L-m_1}) &= -h_L + h_{L-(2K-1)}, \\ \sum_{m_2=2K}^{4K-1} (h_{L-m_2+1} - h_{L-m_2}) &= h_{L-(2K-1)} - h_{L-(4K-1)}, \\ \sum_{m_3=4K}^{6K-1} (-h_{L-m_3+1} + h_{L-m_3}) &= -h_{L-(4K-1)} + h_{L-(6K-1)}.\end{aligned}$$

We can use these to simplify $\phi(Li+1)$:

$$\begin{aligned}\phi(Li+1) &= h_{L+1} - h_L - (-h_L + h_{L-(2K-1)}) + \\ &\quad - [2K^2 + 8K + 3] (h_{L-(2K-1)} - h_{L-(4K-1)}) + \\ &\quad - (-h_{L-(4K-1)} + h_{L-(6K-1)}) \\ &= h_{L+1} - [2K^2 + 8K + 4] (h_{(L+1)-2K} - h_{(L+1)-4K}) - h_{(L+1)-6K}.\end{aligned}$$

Using the linear recurrence in Lemma 3.1.4 we see that $\phi(Li+1) = 0$. Putting both cases together we get $\phi(n) = 0$ for all $n > (6K-1)i+1$. Therefore, the linear recurrence (3.15) does in fact annihilate the sequence produced by (3.13). \square

In the same spirit as Theorem 3.2.4 we can use Theorem 3.2.3 to prove integrality of the sequence produced by

$$b_n b_{n-2Ki} = b_{n-i} b_{n-(2K-1)i} + b_{n-Ki} + b_{n-Ki}. \quad (3.16)$$

Theorem 3.2.6. *Consider the sequence $\{b_n\}_{n=1}^{\infty}$ defined by the recurrence (3.16), and the sequence $\{e_n\}_{n=1}^{\infty}$ defined by the recurrence (3.11). Then*

$$b_{Li+m} = e_{L+1}, \quad \text{for } m \in \{1, \dots, i\}. \quad (3.17)$$

As before, there is a linear annihilator for the sequence produced by (3.16).

Proposition 3.2.7. *If the sequence $\{b_n\}_{n=1}^{\infty}$ is given by the recurrence (3.16) then it is also annihilated by the linear recurrence*

$$\begin{aligned} b_n = & b_{n-1} + \sum_{m_1=1}^{2K-2} (-b_{n-m_1i} + b_{n-(m_1i+1)}) + \\ & + [2K^2 + 6K - 2] \sum_{m_2=2K-1}^{4K-3} (b_{n-m_2i} - b_{n-(m_2i+1)}) + \\ & + \sum_{m_3=4K-2}^{6K-4} (-b_{n-m_3i} + b_{n-(m_3i+1)}) \end{aligned} \quad (3.18)$$

This method of finding linear annihilators of quadratic recurrences is useful for proving integrality of sequences, however it is not an all encompassing technique. Notice that the four linear annihilators, (3.7), (3.12), (3.15), and (3.18), do not fully prove any of the subcases of Conjecture 3.1.6. One might have expected, after seeing the linear recurrences for $\{h_n\}_{n=1}^{\infty}$ and $\{e_n\}_{n=1}^{\infty}$, that all integer sequences produced by (3.10) have linear annihilators with the same general structure. However, we now see that is not the case. They may all have linear annihilators, but the structure of these annihilators is not uniform. In fact, experimentally the structure of the linear annihilator depends on the greatest common divisor of k , i , and j (parameters of (3.10)). Although I do believe (based on experimental observations) that every integer sequence produced by (3.10) has a corresponding linear annihilator, I have not discovered a general form. However, we can still prove the full backwards implication in Conjecture 3.1.6 using Fomin and Zelevinsky's Laurent phenomenon techniques explained in Section 3.1.2.

Before we see this, I will describe the experimental method used to conjecture these linear recurrences.

3.2.1 Conjecturing a Linear Recurrence Using a Hankel Matrix

In Sections 3.1.4 and 3.2 we saw that sequences produced by a specific nonlinear recurrences can be annihilated by related linear recurrences. These linear recurrences were conjectured using linear algebra. Let $\{g_n\}_{n=1}^{\infty}$ be an arbitrary sequence, and consider the following $(m+1) \times (m+1)$ matrix, known as a Hankel matrix [26]:

$$G_{n,m} = \begin{pmatrix} g_n & g_{n+1} & \cdots & g_{n+m} \\ g_{n+1} & g_{n+2} & \cdots & g_{n+1+m} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n+m} & g_{n+m+1} & \cdots & g_{n+2m} \end{pmatrix}.$$

If an $m \in \mathbb{Z}$ can be found such that $\det(G_{n,m}) = 0$ for all n , then there is a linear recurrence for the sequence $\{g_n\}_{n=1}^{\infty}$ of order m . Recall that the columns of a matrix with determinant zero are linearly dependent. By looking at the eigenvectors for the eigenvalue $\lambda = 0$ we can find the linear recurrence explicitly. Consider an eigenvector, $\vec{v} = \langle v_0, \dots, v_m \rangle$, such that $G_{n,m} \cdot \vec{v} = \vec{0}$ for all $n \geq 1$. Then the linear recurrence for $\{g_n\}_{n=1}^{\infty}$ is simply given by

$$v_0 g_n + v_1 g_{n+1} + \cdots + v_m g_{n+m} = 0.$$

To conjecture the linear recurrences in the previous sections I simply used Maple to calculate these $G_{n,m}$ for increasing values of m . When an m was found such that $\det(G_{n,m}) = 0$ for all $n \leq N$ (for some large value of N) I calculated eigenvectors and searched for patterns.

This is a concrete example of how experimental techniques can be used to automatically make conjectures. I have created a Maple package, HANKEL, that implements the above process. The three most useful procedures from HANKEL are:

Xn: Creates terms in the sequence (3.10).

Inputs: Integers n , i , j , and k

Output: The n^{th} term in the sequence produced by the recurrence (3.10).

Try: `[seq(Xn(n,1,2,5),n=1..100)]`;

ConjLinOrder: Conjectures the order of a linear annihilator of a given sequence.

Inputs: A sequence S (in the form of a list) and an integer m .

Output: The conjectured order (less than m) of a linear recurrence that annihilates S . Uses the Hankel matrix technique described above.

Try: `ConjLinOrder([seq(Xn(n,1,2,5),n=1..100)], 13)`;

ConjLinRecur: Conjectures the linear annihilator of a given sequence.

Inputs: A list S (in the form of a list) and an integer m

Outputs: The linear recurrence of order `ConjLinOrder(S,m)` that is conjectured to annihilate S . This will be the eigenvector associated to the zero eigenvalue of the associated Hankel matrix.

Try: `ConjLinRecur([seq(Xn(n,1,1,3),n=1..100)], 13)`;

There is also `ConjLinRecurVerbose` which includes print statements to interpret the output of `ConjLinRecur`. The code can be found on my website.

This technique works well to conjecture linear recurrences that annihilate *specific* sequences, however it does not work well in the general case. If we have a recurrence with parameters, like (3.10) for example, we cannot examine the matrix $G_{n,m}$ and take its determinant unless we consider specific values for the parameters. Because of the conjectures that this technique has been able to make (more than what has been presented in the previous sections), I expect that this technique can eventually be used to prove the backwards implication of Conjecture 3.1.6. However, for now we turn to the Laurent phenomenon for this proof.

3.3 Using the Laurent Phenomenon

In Section 3.1.2 we saw Theorem 3.1.2, which gives sufficient conditions for a recurrence of the form

$$x_n x_{n-k} = F(x_{n-1}, \dots, x_{n-k+1}),$$

with formal variable initial conditions to produce a sequence of Laurent polynomials. In order to apply this theorem we must construct the sequence $\{G_n\}_{n=k-1}^0$, from F . If we find in the end that $G_0 = F$, and if F satisfies some other minimal conditions, then we know that the recurrence produces Laurent polynomials. We will now use this to prove the backwards implication in Conjecture 3.1.6.

Theorem 3.3.1. *Consider the quadratic recurrence (3.10), with the usual initial conditions, $x_i = 1$ for $1 \leq i \leq k$. If*

1. k is even, i is odd, and $j = \frac{k}{2}$, or
2. k is even, i is even, and $j = \frac{i}{2}$, $j = \frac{k}{2}$, or $j = \frac{k-i}{2}$, or
3. k is odd, i is odd, and $j = \frac{k-i}{2}$, or
4. k is odd, i is even, and $j = \frac{i}{2}$,

then $\{x_n\}_{n=1}^\infty$ is an infinite sequence of integers.

Proof. In each of the four cases the proof follows the same general idea, so we will only show case 3, where k is odd, i is odd, and $j = \frac{k-i}{2}$. Let $k := 2K + 1$ and $i := 2I + 1$, from which we have that $j = K - I$. Also, assume that $2I + 1 < k - i = 2K - 2I$ and $K - I < k - j = K + I + 1$. The recurrence (3.10) in this case is

$$x_n x_{n-(2K+1)} = x_{n-(2I+1)} x_{n-(2K-2I)} + x_{n-(K-I)} + x_{n-(K+I+1)}, \quad (3.19)$$

and so

$$F(y_1, \dots, y_{2K}) = y_{2I+1} y_{2K-2I} + y_{K-I} + y_{K+I+1}.$$

Clearly, F satisfies requirement (i) in Theorem 3.1.2 as it is not divisible by any y_n . In addition, we can show that F satisfies (ii) in Theorem 3.1.2. If we replace one of the

terms in F by 0 (which is how we construct Q_n), then we will be left with two terms. The sets of variables for each term are disjoint, so there will be no way to factor the resulting polynomial. To prove (iii), we will have two cases depending on the relative order of i and j . We first assume that $i < j$, so

$$2I + 1 < K - I < K + I + 1 < 2K - 2I. \quad (3.20)$$

We need to show that $G_0 = F$, so we must start with $G_{k-1} = G_{2K} := F$ and use the recursive procedure described in Section 3.1.2 to eventually construct G_0 . In order to find G_{n-1} from G_n the first step is to replace y_n with $\frac{Q_n}{y_n}$ in G_n . So, since the only variables in G_{2K} are y_{2I+1} , y_{2K-2I} , y_{K-I} , and y_{K+I+1} , the first time G_n will change is when $n = 2K - 2I$, the largest of the four indices. So $G_n = G_{2K}$ for $2K - 2I \leq n \leq 2K$. To get $G_{2K-2I-1}$ we first need Q_{2K-2I} .

$$\begin{aligned} Q_{2K-2I} &= F_{2K-2I}|_{y_{2K}=0} \\ &= F(y_{2K-2I+1}, \dots, y_{2K}, 0, y_1, \dots, y_{2K-2I-1}) \\ &= 0 \cdot y_{2K-4I-1} + y_{K-3I-1} + y_{K-I} \\ &= y_{K-3I-1} + y_{K-I}. \end{aligned}$$

Then we can find $\tilde{G}_{2K-2I-1}$,

$$\begin{aligned} \tilde{G}_{2K-2I-1} &= G_{2K-2I}|_{y_{2K-2I} = \frac{Q_{2K-2I}}{y_{2K-2I}}} \\ &= \frac{y_{2I+1}y_{K-3I-1} + y_{2I+1}y_{K-I} + y_{K-I}y_{2K-2I} + y_{K+I+1}y_{2K-2I}}{y_{2K-2I}}. \end{aligned}$$

Now, L is a Laurent monomial which makes $\frac{\tilde{G}_{2K-2I-1}}{L}$ a polynomial in $\mathbb{Z}[y_1, \dots, y_{2K}]$ not divisible by any non-unit in \mathbb{Z} . So $L = y_{2K-2I}^{-1}$, and then $\tilde{\tilde{G}}_{2K-2I-1}$ is

$$\begin{aligned} \tilde{\tilde{G}}_{2K-2I-1} &= \frac{\tilde{G}_{2K-2I-1}}{L} \\ &= y_{2I+1}y_{K-3I-1} + y_{2I+1}y_{K-I} + y_{K-I}y_{2K-2I} + y_{K+I+1}y_{2K-2I}. \end{aligned}$$

The final step in creating $G_{2K-2I-1}$ is finding b , the maximal power of Q_{2K-2I} which

divides $\tilde{G}_{2K-2I-1}$. In this case, $b = 0$, and so

$$\begin{aligned} G_{2K-2I-1} &= \frac{\tilde{G}_{2K-2I-1}}{Q_{2K-2I}^b} = \tilde{G}_{2K-2I-1} \\ &= y_{2I+1}y_{K-3I-1} + y_{2I+1}y_{K-I} + y_{K-I}y_{2K-2I} + y_{K+I+1}y_{2K-2I}. \end{aligned}$$

Now, to find the index for the next new G_n we must look at the indices of the terms that are in $G_{2K-2I-1}$ and find the largest that is less than $2K - 2I - 1$. The indices in $G_{2K-2I-1}$ are

$$2I + 1, K - 3I - 1, K - I, 2K - 2I, K + I + 1$$

and, referring back to (3.20), the largest that is less than $2K - 2I - 1$ is $K + I + 1$. Therefore, $G_n = G_{2K-2I-1}$ for $K + I + 1 \leq n \leq 2K - 2I - 1$. For the rest of this proof I will omit the discussion for creating Q , \tilde{G} , \tilde{G} , and G , as we have already seen these once and they all follow the same steps.

$$\begin{aligned} Q_{K+I+1} &= F_{K+I+1}|_{y_{2K}=0} \\ &= F(y_{K+I+2}, \dots, y_{2K}, 0, y_1, \dots, y_{K+I}) \\ &= y_{K+3I+2}y_{K-I} + y_{2I+1}. \end{aligned}$$

$$\begin{aligned} \tilde{G}_{K+I} &= G_{K+I+1}|_{y_{K+I+1} = \frac{Q_{K+I+1}}{y_{K+I+1}}} \\ &= \frac{y_{2I+1}y_{K-3I-1}y_{K+I+1} + y_{2I+1}y_{K-I}y_{K+I+1} + y_{K-I}y_{2K-2I}y_{K+I+1}}{y_{K+I+1}} + \\ &\quad + \frac{y_{2K-2I}y_{K+3I+2}y_{K-I} + y_{2I+1}y_{2K-2I}}{y_{K+I+1}}. \end{aligned}$$

$$\begin{aligned} \tilde{\tilde{G}}_{K+I} &= \frac{\tilde{G}_{K+I}}{L} = \frac{\tilde{G}_{K+1}}{(y_{K+I+1})^{-1}} \\ &= y_{2I+1}y_{K-3I-1}y_{K+I+1} + y_{2I+1}y_{K-I}y_{K+I+1} + y_{K-I}y_{2K-2I}y_{K+I+1} + \\ &\quad + y_{2K-2I}y_{K+3I+2}y_{K-I} + y_{2I+1}y_{2K-2I}. \end{aligned}$$

$$\begin{aligned} G_{K+I} &= \frac{\tilde{\tilde{G}}_{K+I}}{Q_{K+I+1}^b} = \tilde{\tilde{G}}_{K+I} \\ &= y_{2I+1}y_{K-3I-1}y_{K+I+1} + y_{2I+1}y_{K-I}y_{K+I+1} + y_{K-I}y_{2K-2I}y_{K+I+1} + \\ &\quad + y_{2K-2I}y_{K+3I+2}y_{K-I} + y_{2I+1}y_{2K-2I}. \end{aligned}$$

Once again, we must find the next highest index out of those in G_{K+I} . The indices are

$$2I + 1, K - 3I - 1, K + I + 1, K - I, 2K - 2I, K + 3I + 2$$

and, again we refer to (3.20) in order to see that the highest that is less than $K + I$ is $K - I$. So $G_n = G_{K+I}$ for $K - I \leq n \leq K + I$.

$$\begin{aligned} Q_{K-I} &= F_{K-I}|_{y_{2K}=0} \\ &= F(y_{K-I+1}, \dots, y_{2K}, 0, y_1, \dots, y_{K-I-1}) \\ &= y_{K-3I-1}y_{K+I+1} + y_{2K-2I}. \end{aligned}$$

$$\begin{aligned} \tilde{G}_{K-I-1} &= G_{K-I}|_{y_{K-I}=\frac{Q_{K-I}}{y_{K-I}}} \\ &= \frac{(y_{K+I+1}y_{2K-2I} + y_{2K-2I}y_{K+3I+2} + y_{2I+1}y_{K+I+1} + y_{2I+1}y_{K-I})}{y_{K-I}} \\ &\quad \cdot \frac{(y_{K-3I-1}y_{K+I+1} + y_{2K-2I})}{y_{K-I}}. \end{aligned}$$

$$\begin{aligned} \tilde{\tilde{G}}_{K-I-1} &= \frac{\tilde{G}_{K-I-1}}{L} = \frac{\tilde{G}_{K-I-1}}{(y_{K-1})^{-1}} \\ &= (y_{K+I+1}y_{2K-2I} + y_{2K-2I}y_{K+3I+2} + y_{2I+1}y_{K+I+1} + y_{2I+1}y_{K-I}) \\ &\quad \cdot (y_{K-3I-1}y_{K+I+1} + y_{2K-2I}). \end{aligned}$$

$$\begin{aligned} G_{K-I-1} &= \frac{\tilde{\tilde{G}}_{K-I-1}}{Q_{K-I}^b} = \frac{\tilde{\tilde{G}}_{K-I-1}}{Q_{K-I}^1} \\ &= y_{K+I+1}y_{2K-2I} + y_{2K-2I}y_{K+3I+2} + y_{2I+1}y_{K+I+1} + y_{2I+1}y_{K-I}. \end{aligned}$$

Once more, we find that the next highest index less than $K - I - 1$ out of those in G_{K-I-1} is $2I + 1$. So $G_n = G_{K-I-1}$ for $2I + 1 \leq n \leq K - I - 1$.

$$\begin{aligned} Q_{2I+1} &= F_{2I+1}|_{y_{2K}=0} \\ &= F(y_{2I+2}, \dots, y_{2K}, 0, y_1, \dots, y_{2I}) \\ &= y_{K+I+1} + y_{K+3I+2}. \end{aligned}$$

$$\begin{aligned}\tilde{G}_{2I} &= G_{2I+1} \Big|_{y_{2I+1} = \frac{Q_{2I+1}}{y_{2I+1}}} \\ &= \frac{(y_{K+I+1} + y_{K+3I+2})(y_{2I+1}y_{2K-2I} + y_{K-I} + y_{K+I+1})}{y_{2I+1}}.\end{aligned}$$

$$\begin{aligned}\tilde{\tilde{G}}_{2I} &= \frac{\tilde{G}_{2I}}{L} = \frac{\tilde{G}_{2I}}{y_{2I+1}^{-1}} \\ &= (y_{K+I+1} + y_{K+3I+2})(y_{2I+1}y_{2K-2I} + y_{K-I} + y_{K+I+1}).\end{aligned}$$

$$\begin{aligned}G_{2I} &= \frac{\tilde{\tilde{G}}_{2I}}{Q_{2I+1}^b} = \frac{\tilde{\tilde{G}}_{2I}}{Q_{2I+1}^1} \\ &= y_{2I+1}y_{2K-2I} + y_{K-I} + y_{K+I+1}\end{aligned}$$

All of the indices in G_{2I} are greater than $2I$, so $G_n = G_{2I}$ for $0 \leq n \leq 2I$. Therefore, $G_0 = G_{2I} = y_{2I+1}y_{2K-2I} + y_{K-I} + y_{K+I+1}$, and this is precisely F . So F satisfies requirement (iii).

So far in this proof we have only covered the case when k is odd, i is odd, $j = \frac{k-i}{2}$, and $i < j$. In order to see the case when $i > j$ we again follow the procedure outlined in Section 3.1.2 to recursively create the polynomials G_n and see that, indeed $G_0 = F$. Once we have completed case 3 we again follow Fomin and Zelevinsky's procedure for cases 1, 2, and 4. In all cases we see that $G_0 = F$ and so the theorem follows. \square

3.4 Conclusion

We began in this chapter by introducing the Somos- k recurrences. We gave three different techniques for proving that Somos- k (for $k = 4, 5, 6$, and 7) produces an integer sequence: number theory, the Laurent phenomenon and cluster algebras, and enumeration. Having seen that the Somos- k recurrences are completely understood we turned to recurrences of a similar form, that still possess the integrality property. Many of these recurrences are homogeneous in degree, as Somos- k is; we then saw that this need not be the case. We spent the rest of the chapter studying the three parameter recurrence (3.10). In [17], Paul Heideman and I proved that the special case where $k = 2K + 1$, $i = 1$, and $j = K$, produces an integer sequence. Instead of using

one of the three proof methods mentioned above, we showed that the sequence also satisfied a linear recurrence. I then used this linear recurrence technique to establish integrality for three additional sub-cases of (3.10): first, $k = 2K$, $i = 1$, and $j = K$; then, $k = (2K + 1)i$ and $j = Ki$; finally $k = 2Ki$ and $j = Ki$.

Though the method of finding linear annihilators does give us a tangible reason for integrality, it has not proved to be a unifying technique. Instead, we proved integrality of sequences produced by (3.10) (when k , i , and j satisfy some conditions) using the Laurent phenomenon technique. This settled one direction of a conjecture put forth by Heideman and me in [17].

Chapter 4

Nonlinear Recurrences that Unexpectedly Generate Rationals

In the previous chapter we explored a family of nonlinear recurrences, inspired by the Somos sequences, that produce integer sequences when rational numbers were expected. The general form of these recurrences was

$$x_n x_{n-k} = P(x_{n-1}, \dots, x_{n-k+1}), \quad (4.1)$$

where P is a polynomial. This is just a special case of the more general form

$$x_n = R(x_{n-1}, \dots, x_{n-k}), \quad (4.2)$$

where R is a rational function (recall that recurrences of this form were studied in Chapter 2 under the name *difference equations*, see Definition 2.1.1). We get (4.1) when we let $R = \frac{P}{x_{n-k}}$ in (4.2). In this chapter we will further generalize (4.2) and study “recurrences” of the form

$$x_n^m = F(x_{n-1}, \dots, x_{n-k}), \quad \text{for some } 1 < m \in \mathbb{N}, \quad (4.3)$$

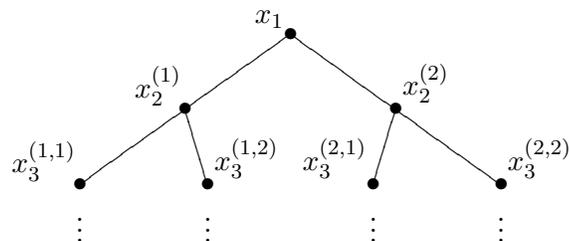
which we will call *m-recurrences*.

There are two major differences between (4.2) and (4.3). First, we raise x_n to an integer power greater than 1. Secondly, we allow x_n , the next term we are interested in finding, to be on the right side of the equation. Equivalently, we can write an m -recurrence as

$$\sum_{i=0}^m F_i(x_{n-1}, \dots, x_{n-k}) x_n^i = 0. \quad (4.4)$$

In this chapter we will take F_i to be polynomials, but in general they could be any functions $F_i : \mathbb{C} \rightarrow \mathbb{C}$. Written in this equivalent form we see that in order to compute

x_n we must solve a degree m equation. Thus, we will expect to produce complex numbers when the initial conditions are in \mathbb{Z} or \mathbb{Q} . Moreover, when we solve this degree- m equation for x_n we get m solutions (counting multiplicities). So instead of producing a single sequence, as is the usual case, we may produce infinitely many sequences from one set of initial conditions. We will consider the set of sequences as being stored as a tree, which I will refer to as a *sequence tree*. Given specific x_{n-k}, \dots, x_{n-1} there are m possibilities for x_n which will be the children of x_{n-1} in the sequence tree. When $m = 2$ and $k = 1$ we produce the following sequence tree:



The superscripts refer to the path from the root to x_n through the sequence tree. For example, to get to $x_3^{(2,1)}$ from x_1 , take the second child of the root to get to $x_2^{(2)}$, and then the first child of $x_2^{(2)}$.

m -Recurrences in which $k = 1$ are also known as $m - l$ *correspondences* where l is the degree of x_{n-1} in F [37]. For a given value of x_{n-1} there are m possibilities, or m images, for x_n . Conversely, for a given value of x_n there are l possibilities (or pre-images) for x_{n-1} that could have produced x_n .

One way to construct recurrences that obviously generate rational numbers is to begin with a recurrence that generates rational numbers and then find a recurrence for its sequence of ratios. Given a sequence $\{x_n\}_{n=1}^{\infty}$, we will call $\left\{ \frac{x_{n+1}}{x_n} \right\}_{n=1}^{\infty}$ the *sequence of ratios of $\{x_n\}$* . Obviously, if a sequence $\{x_n\}_{n=1}^{\infty}$ consists of rational numbers then the sequence of ratios of $\{x_n\}$ is also rational. Of course, it may not be the case that the obvious recurrence for the sequence of ratios is an m -recurrence. In particular, this is not the case for Somos-4, a sequence of great interest in the upcoming discussion. However we may be able to find an alternate recurrence that is of this desired form, as we will see in the case of Somos-4. We can then generalize and find new recurrences that were not specifically constructed to generate rational numbers. This is what we

will do now, first for Somos-4 and then in general.

4.1 A 2-Recurrence of Order 1 That Generates Rational Numbers

Consider the *generalized* Somos-4 recurrence, a special case of the three-term Gale-Robinson recurrence (3.3):

$$s_n s_{n-4} = \alpha s_{n-1} s_{n-3} + \beta s_{n-2}^2, \quad \alpha, \beta \in \mathbb{Z}, \quad (4.5)$$

with integer initial conditions. When $\alpha = \beta = 1$, and the initial conditions are $s_i = 1$ for $1 \leq i \leq 4$, we simply get the Somos-4 recurrence (3.2). With the intention to construct an m -recurrence, we consider the sequence of ratios of $\{s_n\}$, $\{t_n\}_{n=1}^\infty$, and then the sequence of ratios of $\{t_n\}$, $\{f_n\}_{n=1}^\infty$, defined by

$$t_n = \frac{s_{n+1}}{s_n}, \quad \text{and} \quad f_n = \frac{t_{n+1}}{t_n} = \frac{s_{n+2}s_n}{s_{n+1}^2}.$$

Since $\{s_n\}_{n=1}^\infty$ is clearly a rational sequence (in fact it is an integer sequence when the initial conditions are uniformly 1, see [11]) we know that $\{t_n\}_{n=1}^\infty$ and $\{f_n\}_{n=1}^\infty$ are rational sequences. The following proposition gives us a 2-recurrence for f_n which we will study, and eventually generalize.

Proposition 4.1.1. *The following 2-recurrence annihilates the sequence $\{f_n\}_{n=1}^\infty$:*

$$f_n^2 f_{n+1}^2 + [\alpha - (2\alpha + \beta + 1)f_n] f_{n+1} + \alpha f_n + \beta = 0. \quad (4.6)$$

Notice that this 2-recurrence is written in the form (4.4).

In [18, 19, 20, 36], Hone and Swart study an expression equivalent to (4.6) from the perspective of elliptic functions. Whereas our goal is to generalize the idea of a recurrence, their main goal is to produce explicit formulas, and asymptotics, for the generalized Somos sequences in terms of Weierstrass sigma functions. In addition, they obtain a sufficient condition for (4.5) to produce integers when the Laurent phenomenon doesn't apply (e.g., when the initial conditions are integers, but not all equal to 1). The following theorem, which appears in [20] in a slightly different form, contains an expression equivalent to (4.6), and will aid us in proving Proposition 4.1.1.

Theorem 4.1.2 (Hone, Swart 2008). *If $\{s_n\}_{n=1}^{\infty}$ is a generalized Somos-4 sequence, defined by recurrence (4.5), with coefficients α and β , then the quantity*

$$\begin{aligned} S &= \frac{s_n s_{n+3}}{s_{n+1} s_{n+2}} + \alpha \left(\frac{(s_{n+1})^2}{s_n s_{n+2}} + \frac{(s_{n+2})^2}{s_{n+1} s_{n+3}} \right) + \frac{\beta s_{n+1} s_{n+2}}{s_n s_{n+3}} \\ &= f_n f_{n+1} + \alpha \left(\frac{1}{f_n} + \frac{1}{f_{n+1}} \right) + \frac{\beta}{f_n f_{n+1}} \end{aligned} \quad (4.7)$$

is independent of n .

Proof of Proposition 4.1.1. Since S , from Theorem 4.1.2, is independent of n , we can compute it by simply considering the case when $n = 1$. We have that

$$\begin{aligned} S &= f_1 f_2 + \alpha \left(\frac{1}{f_1} + \frac{1}{f_2} \right) + \frac{\beta}{f_1 f_2} \\ &= \frac{s_3 s_1}{s_2^2} \frac{s_4 s_2}{s_3^2} + \alpha \left(\frac{s_2^2}{s_3 s_1} + \frac{s_3^2}{s_4 s_2} \right) + \frac{\beta s_2^2 s_3^2}{s_3 s_1 s_4 s_2} \\ &= 1 + 2\alpha + \beta. \end{aligned}$$

Now consider the following quantity

$$T = f_n f_{n+1} (S - (2\alpha + \beta + 1)) = 0.$$

Since $T = 0$ this indicates a possible homogeneous recurrence for f_n (i.e., a recurrence with no constant term). To see this we write T in terms of f_n and f_{n+1} using the definition of S from Theorem 4.1.2:

$$\begin{aligned} T &= f_n f_{n+1} (S - (2\alpha + \beta + 1)) = \\ &= f_n f_{n+1} \left(f_n f_{n+1} + \alpha \left(\frac{1}{f_n} + \frac{1}{f_{n+1}} \right) + \frac{\beta}{f_n f_{n+1}} - (2\alpha + \beta + 1) \right) \\ &= f_n^2 f_{n+1}^2 + (\alpha - (2\alpha + \beta + 1) f_n) f_{n+1} + \alpha f_n + \beta = 0. \end{aligned}$$

Thus we see that (4.6) annihilates the sequence $\{f_n\}_{n=1}^{\infty}$. Rewriting, we can see that (4.6) is a 2-recurrence with $k = 1$:

$$f_{n+1}^2 = \frac{(2\alpha + \beta + 1) f_{n+1} f_n - \alpha f_{n+1} - \alpha f_n - \beta}{f_n^2}.$$

□

The sequence tree for the 2-recurrence (4.6), where $\alpha = \beta = 1$ and $f_1 = 1$, is shown in Figure 4.1 and appears to consist of only rational numbers. From Proposition 4.1.1 we know that (4.6) annihilates a sequence of rational numbers, namely $\{f_n\}_{n=1}^\infty$. However, one might wonder whether *only* rational numbers are produced when we start with $f_1 = 1$ and repeatedly apply the 2-recurrence to create the sequence tree. Could there be a branch in which complex numbers appear? The sequence $\left\{\frac{s_{n+2}s_n}{s_{n+1}^2}\right\}_{n=1}^\infty$ occurs as a branch of the sequence tree, and it turns out that the sequence tree does consist of only rational numbers. We will see why this is true through a generalization of (4.6).

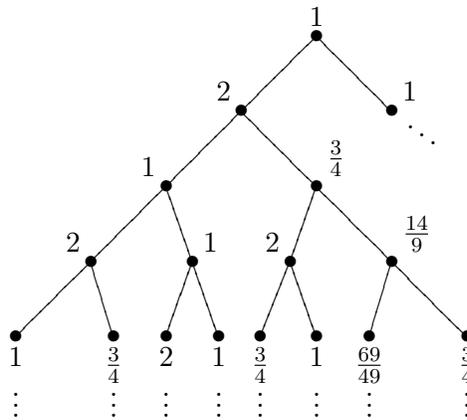


Figure 4.1: The sequence tree for (4.6) with $\alpha = \beta = 1$

Recall the form of an m -recurrence, $\sum_{i=0}^m F_i(x_{n-1}, \dots, x_{n-k})x_n^i = 0$, as given in (4.4). The recurrence (4.6) is of this type with $k = 1$, $m = 2$, and

$$F_2(Y) = Y^2,$$

$$F_1(Y) = -(2\alpha + \beta + 1)Y + \alpha,$$

$$F_0(Y) = \alpha Y + \beta.$$

One way to generalize this is to look at m -recurrences of the form (4.4) with $k = 1$, $m = 2$, and

$$F_2(Y) = Y^2 + A_1Y + A_0$$

$$F_1(Y) = B_2Y^2 + B_1Y + B_0$$

$$F_0(Y) = C_2Y^2 + C_1Y + C_0.$$

We can prove some minimal sufficient conditions for (4.4), with the above expressions for F_0, F_1, F_2 , to produce a rational sequence tree.

Proposition 4.1.3. *Let $X := x_n$ and $Y := x_{n-1}$. Consider the 2-recurrence*

$$(Y^2 + A_1Y + A_0)X^2 + (B_2Y^2 + B_1Y + B_0)X + (C_2Y^2 + C_1Y + C_0) = 0, \quad (4.8)$$

with initial condition $x_1 = 1$. The corresponding sequence tree is rational if

$$(i) \quad A_1 = B_2, \quad A_0 = C_2, \quad B_0 = C_1, \quad \text{and}$$

$$(ii) \quad (B_2 + B_1 + B_0)^2 - 4(A_1 + A_0 + 1)(C_2 + C_1 + C_0) = q^2 \quad \text{for some } q \in \mathbb{Q}.$$

The 2-recurrence (4.8) is much more general than the 2-recurrence for the ratios of ratios of the generalized Somos-4 sequence, (4.6). For example, let $A_0 = C_2 = 1$, $A_1 = B_2 = -1$, $B_0 = C_1 = 1$, $B_1 = -4$, and $C_0 = 1$, then the quantity from condition (ii) is

$$\begin{aligned} (B_2 + B_1 + B_0)^2 - 4(A_1 + A_0 + 1)(C_2 + C_1 + C_0) &= \\ &= (-1 - 4 + 1)^2 - 4(-1 + 1 + 1)(1 + 1 + 1) \\ &= 16 - 12 = 4 = 2^2. \end{aligned}$$

Since this is the square of a rational number, the 2-recurrence,

$$X^2Y^2 - XY(X + Y) + (X^2 + Y^2) - 4XY + (X + Y) + 1 = 0,$$

where $X := x_n$ and $Y := x_{n-1}$, produces a rational sequence tree. This is clearly not a special case of the 2-recurrence for the sequence of ratios of ratios of the generalized Somos-4 (4.6). Therefore, once proved, Proposition 4.1.3 gives us a family of 2-recurrences with $k = 1$ that produce rational sequence trees even though they were not specially constructed to do so.

Proof of Proposition 4.1.3. First, make the substitutions in (i), and define $P(X, Y)$:

$$\begin{aligned} P(X, Y) &:= (Y^2 + A_1Y + A_0)X^2 + (A_1Y^2 + B_1Y + B_0)X + (A_0Y^2 + B_0Y + C_0) \\ &= X^2Y^2 + A_1XY(X + Y) + A_0(X^2 + Y^2) + B_1XY + B_0(X + Y) + C_0. \end{aligned}$$

This is known as an *Euler-Chasles correspondence* (see [2, 37]) and is well known in the theory of algebraic correspondences to have nice properties. Notice that $P(X, Y)$ is symmetric in X and Y ; this is the key fact going forward.

This proof will be accomplished by induction on n : assuming that $x_{n-2}, x_{n-1} \in \mathbb{Q}$, we will show that $x_n \in \mathbb{Q}$. We are given that $x_1 = 1 \in \mathbb{Q}$, so to complete the base case we must show that $x_2 \in \mathbb{Q}$. Notice that the expression in condition (ii) is the discriminant of the quadratic equation $P(X, 1) = 0$. Since this discriminant is the square of a rational number we know that $x_2 \in \mathbb{Q}$. Thus the base case, $x_n \in \mathbb{Q}$ for $n = 1, 2$, is satisfied.

Assume, as the induction hypothesis, that x_{n-2} and its two children, $x_{n-1}^{(1)}$ and $x_{n-1}^{(2)}$ are rational. These children of x_{n-2} are the solutions of $P(X, x_{n-2}) = 0$:

$$\{x_{n-1}^{(1)}, x_{n-1}^{(2)}\} = \{X \in \mathbb{R} : P(X, x_{n-2}) = 0\}.$$

Then, since P is symmetric in X and Y , we know that $P(x_{n-2}, x_{n-1}^{(i)}) = 0$ for $i = 1, 2$. The values for x_n , the descendant of a specific x_{n-2} through $x_{n-1}^{(i)}$, are the two elements of the set $\{X : P(X, x_{n-1}^{(i)}) = 0\}$. But of course, one of the elements of this set is x_{n-2} , which we have assumed is rational. Since we are solving a quadratic equation with rational coefficients and we know that one of the roots is rational, it must be that the other root is rational as well.

From the base case, $x_1, x_2^{(1)}, x_2^{(2)} \in \mathbb{Q}$, and the inductive step, we clearly see that $x_3^{(i,j)} \in \mathbb{Q}$ for $i, j \in \{1, 2\}$. Then continuing by induction we see that the entire sequence tree is rational. \square

In the next section we will use this 2-recurrence of order 1 (4.8) to obtain a family of higher order 2-recurrences.

4.2 Higher Order 2-Recurrences that Surprisingly Generate Rationals

In the previous section we looked at 2-recurrences of order 1, also known as 2-2 correspondences. These turn out to have many connections to elliptic curves and are well studied from that perspective. We now turn our attention to higher order 2-recurrences. As before, we will first explore in the context of Somos-4, then look into generalizations.

Recall the 2-recurrence (4.6) for the ratios of ratios of generalized Somos-4 sequence:

$$f_{n+1}^2 f_n^2 - (2\alpha + \beta + 1)f_{n+1}f_n + \alpha f_{n+1} + \alpha f_n + \beta = 0,$$

where $f_n = \frac{s_{n+2}s_n}{s_{n+1}^2}$, and s_n is given by the generalized Somos-4 recurrence. By substituting the definition of f_n into (4.6), we get a 2-recurrence of order 3 for the generalized Somos-4 sequence:

$$s_n^2 s_{n+3}^2 + (\alpha s_{n+1}^3 - (2\alpha + \beta + 1)s_{n+2}s_{n+1}s_n) s_{n+3} + \alpha s_{n+2}^3 s_n + \beta s_{n+2}^2 s_{n+1}^2 = 0. \quad (4.9)$$

Just as the 2-recurrence for f_n appeared in Hone and Swart's paper [20], this recurrence for s_n appears as a Diophantine equation in the following theorem first found in [20].

Theorem 4.2.1 (Hone, Swart 2008). *Given $\alpha, \beta \in \mathbb{Z}$ and $S \in \mathbb{Q}$ with $\beta S \in \mathbb{Z}$, suppose that the quartic equation*

$$s^2 v^2 + \alpha(su^3 + t^3 v) + \beta t^2 u^2 = Sstuv \quad (4.10)$$

has a solution of the form $(s, t, u, v) = (A_1, A_2, A_3, A_4)$, with $A_1 = \pm 1$ and non-zero $A_2, A_3, A_4 \in \mathbb{Z}$. Then, provided that the orbit of this set of initial data under (4.5) is non-periodic, it produces infinitely many integer solutions of the Diophantine equation (4.10).

Notice that this applies in the more general setting when the integer initial conditions s_1, \dots, s_4 are not uniformly equal to 1; in this case, S may not equal $2\alpha + \beta + 1$. This theorem states that every set of four consecutive terms in the generalized Somos-4 sequence, produced by (4.5), satisfies the Diophantine equation (4.10) (where $s = s_n$, $t = s_{n+1}$, $u = s_{n+2}$, and $v = s_{n+3}$). However, as we will see later in this section, there are other integer sequences which produce infinitely many solutions to (4.10) and do not satisfy the order 4 recurrence (4.5) for the generalized Somos-4 sequence. In addition we will see an *octic* (total degree 8) Diophantine equation which is satisfied by consecutive terms in a sequence produced by a 2-recurrence analogous to (4.9).

Before we generalize the 2-recurrence of order 3 for Somos-4 let us explore the sequence tree, T_s shown in Figure 4.2, generated by (4.9) with $\alpha = \beta = 1$ and initial conditions $s_1 = s_2 = s_3 = 1$. There are a few things worth noting about T_s . First and

most conspicuously, there are non-integer rational numbers in T_s . However, this is not entirely surprising because we can generally express s_{n+1} in terms of f_n as follows:

$$s_{n+1} = \frac{s_2^n}{s_1^{n-1}} (f_1^{n-1} f_2^{n-2} \cdots f_{n-2}^2 f_{n-1}).$$

Since typically, $f_i \notin \mathbb{Z}$, and not every branch of the sequence tree for f_n equals the sequence of ratios of ratios of s_n , we should not expect that s_{n+1} as given by the recurrence (4.9) would always be an integer.

The next thing to notice about T_s are some interesting branches. The sequence $2^{g(n)}$, where $g(n) = \lfloor \frac{(n-3)^2}{4} \rfloor$, appears as a branch, as well as simply 2^{n-3} . Both are easy to prove by showing that (4.9) annihilates the sequences $s_n = 2^{g(n)}$ and $s_n = 2^{n-3}$, and seeing that their first 3 terms occur consecutively within the initial levels of T_s . The existence of each of these branches will be proved in a more general setting by Theorems 4.2.2 and 4.2.4. It is also worth noting that these exponential sequences, that appear when we reduced the order of the Somos-4 recurrence from 4 to 3, are not annihilated by the order 4 recurrence. In contrast with the formula for the usual s_n in terms of the Weierstrass sigma function associated to a particular elliptic curve [18], these new sequences have closed form formulas in terms of elementary functions. In addition, we get a reduction in the asymptotics. It is known that the original Somos-4 sequence, which of course occurs as a branch in T_s , is asymptotically ϕ^{n^2} for some ϕ which can be found in terms of elliptic curves [18]. Not only do we see a branch that is asymptotically 2^{n^2} , but we also have a branch that grows like 2^n which is a major asymptotic reduction.

We now continue by generalizing (4.9), using the 2-recurrence (4.8) from the previous section, and then proving our observations from the previous paragraph for this more general 2-recurrence. Recall that (4.9) was obtained from (4.6) by replacing f_n by its definition in terms of s_n . Even though (4.8) encompasses more than just those recurrences created as ratios of ratios of integer sequences, we can nevertheless substitute

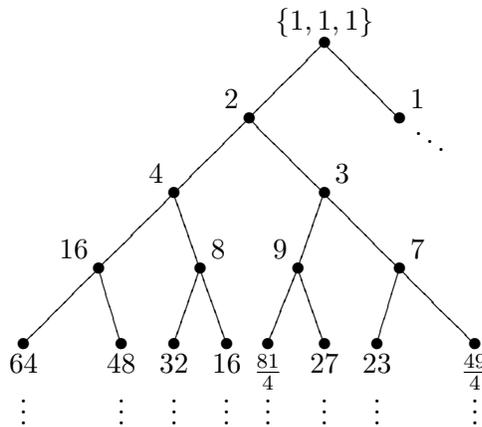


Figure 4.2: The sequence tree for (4.9)

$x_n = \frac{a_{n+2}a_n}{a_{n+1}^2}$ and study the resulting 2-recurrence of order 3:

$$\begin{aligned}
 & (a_n^2 a_{n+2}^2 + A_1 a_n a_{n+1}^2 a_{n+2} + A_0 a_{n+1}^4) a_{n+1}^2 a_{n+3}^2 + \\
 & + (A_1 a_n^2 a_{n+2}^2 + B_1 a_n a_{n+1}^2 a_{n+2} + B_0 a_{n+1}^4) a_{n+1} a_{n+2}^2 a_{n+3} + \\
 & + (A_0 a_n^2 a_{n+2}^2 + B_0 a_n a_{n+1}^2 a_{n+2} + C_0 a_{n+1}^4) a_{n+2}^4 = 0.
 \end{aligned} \tag{4.11}$$

Of course we must satisfy the conditions in Proposition 4.1.3 in order for (4.11) to produce rational numbers. Notice that we get (4.9) from (4.11) if we let $A_1 = A_0 = 0$, $B_1 = -(2\alpha + \beta + 1)$, $B_0 = \alpha$, $C_0 = \beta$, and divide out by the superfluous $a_{n+2}^2 a_{n+1}^2$. As in (4.9), we can find some exponential branches in the sequence tree for (4.11). The following two theorems summarize the types of exponential branches found, which will explain the $2^{\lfloor (n-3)^2/4 \rfloor}$ and 2^{n-3} branches in T_s .

Theorem 4.2.2. *The sequence $a_n = \gamma^{\lfloor n^2/4 \rfloor}$ is annihilated by (4.11) iff γ is a solution to the following quadratic equation*

$$(A_1 + A_0 + 1)\gamma^2 + (A_1 + B_1 + B_0)\gamma + A_0 + B_0 + C_0 = 0.$$

Proof. First note that the value of $\lfloor n^2/4 \rfloor$ depends on the value of n modulo 4.

$$\begin{aligned} n = 4m &\implies \left\lfloor \frac{(4m)^2}{4} \right\rfloor = 4m^2 \\ n = 4m + 1 &\implies \left\lfloor \frac{(4m+1)^2}{4} \right\rfloor = \left\lfloor \frac{16m^2 + 8m + 1}{4} \right\rfloor = 4m^2 + 2m \\ n = 4m + 2 &\implies \left\lfloor \frac{(4m+2)^2}{4} \right\rfloor = \left\lfloor \frac{16m^2 + 16m + 4}{4} \right\rfloor = 4m^2 + 4m + 1 \\ n = 4m + 3 &\implies \left\lfloor \frac{(4m+3)^2}{4} \right\rfloor = \left\lfloor \frac{16m^2 + 24m + 9}{4} \right\rfloor = 4m^2 + 6m + 2 \end{aligned}$$

Now, assume that $a_n = \gamma^{\lfloor n^2/4 \rfloor}$ is annihilated by (4.11) and let $n \equiv 0 \pmod{4}$. By substituting

$$a_n = \gamma^{4m^2}, \quad a_{n+1} = \gamma^{4m^2+2m}, \quad a_{n+2} = \gamma^{4m^2+4m+1}, \quad \text{and} \quad a_{n+3} = \gamma^{4m^2+6m+2}$$

into (4.11) we get:

$$\begin{aligned} &\left[\left(\gamma^{4m^2+4m+1} \right)^2 \left(\gamma^{4m^2} \right)^2 + A_1 \gamma^{4m^2+4m+1} \left(\gamma^{4m^2+2m} \right)^2 \gamma^{4m^2} + \right. \\ &+ A_0 \left(\gamma^{4m^2+2m} \right)^4 \left. \left(\gamma^{4m^2+2m} \right)^2 \left(\gamma^{4m^2+6m+2} \right)^2 + \right. \\ &+ \left[A_1 \left(\gamma^{4m^2+4m+1} \right)^2 \left(\gamma^{4m^2} \right)^2 + B_1 \gamma^{4m^2+4m+1} \left(\gamma^{4m^2+2m} \right)^2 \gamma^{4m^2} + \right. \\ &+ B_0 \left(\gamma^{4m^2+2m} \right)^4 \left. \right] \gamma^{4m^2+2m} \left(\gamma^{4m^2+4m+1} \right)^2 \gamma^{4m^2+6m+2} + \\ &+ \left[A_0 \left(\gamma^{4m^2+4m+1} \right)^2 \left(\gamma^{4m^2} \right)^2 + B_0 \gamma^{4m^2+4m+1} \left(\gamma^{4m^2+2m} \right)^2 \gamma^{4m^2} + \right. \\ &+ C_0 \left(\gamma^{4m^2+2m} \right)^4 \left. \right] \left(\gamma^{4m^2+4m+1} \right)^4 = 0. \end{aligned}$$

Simplify the products of γ 's to get:

$$\begin{aligned} &\left[\gamma^{16m^2+8m+2} + A_1 \gamma^{16m^2+8m+1} + A_0 \gamma^{16m^2+8m} \right] \gamma^{16m^2+16m+4} + \\ &+ \left[A_1 \gamma^{16m^2+8m+2} + B_1 \gamma^{16m^2+8m+1} + B_0 \gamma^{16m^2+8m} \right] \gamma^{16m^2+16m+4} + \\ &+ \left[A_0 \gamma^{16m^2+8m+2} + B_0 \gamma^{16m^2+8m+1} + C_0 \gamma^{16m^2+8m} \right] \gamma^{16m^2+16m+4} = 0. \end{aligned}$$

Now, divide both sides by $\gamma^{32m^2+24m+4}$:

$$\left[\gamma^2 + A_1 \gamma + A_0 \right] + \left[A_1 \gamma^2 + B_1 \gamma + B_0 \right] + \left[A_0 \gamma^2 + B_0 \gamma + C_0 \right] = 0.$$

Finally, collect like terms to get the quadratic equation in γ from the statement of the theorem:

$$[A_1 + A_0 + 1]\gamma^2 + [A_1 + B_1 + B_0]\gamma + [A_0 + B_0 + C_0] = 0.$$

The same procedure as above, carried out with $n \equiv 1 \pmod{4}$, $2 \pmod{4}$, or $3 \pmod{4}$, will yield the same quadratic equation. So, if $\gamma^{\lfloor n^2/4 \rfloor}$ is annihilated by (4.11), then γ must be a solution to this quadratic equation. Similarly, if we assume that γ satisfies this quadratic equation, we see that $\gamma^{\lfloor n^2/4 \rfloor}$ is annihilated by (4.11). \square

Corollary 4.2.3. *If γ satisfies*

$$\gamma^2 + (\alpha - (2\alpha + \beta + 1))\gamma + \alpha + \beta = 0,$$

then $s_n = \gamma^{\lfloor n^2/4 \rfloor}$ is annihilated by the order 3 recurrence (4.9) for the generalized Somos-4 sequence,

$$s_n^2 s_{n+3}^2 + (\alpha s_{n+1}^3 - (2\alpha + \beta + 1)s_{n+2}s_{n+1}s_n) s_{n+3} + \alpha s_{n+2}^3 s_n + \beta s_{n+2}^2 s_{n+1}^2 = 0.$$

In particular, $(\alpha + \beta)^{\lfloor n^2/4 \rfloor}$ is annihilated by (4.9).

Proof. Since (4.9) is obtained by letting $A_1 = A_0 = 0$, $B_1 = -(2\alpha + \beta + 1)$, $B_0 = \alpha$, and $C_0 = \beta$, in the recurrence (4.11) we can apply Theorem 4.2.2. Making these substitutions into the quadratic equation from the statement of Theorem 4.2.2 yields

$$\gamma^2 + (\alpha - (2\alpha + \beta + 1))\gamma + \alpha + \beta = 0$$

as needed. \square

This corollary explains the existence of the $2^{\lfloor (n-3)^2/4 \rfloor}$ branch in the sequence tree for the original Somos-4, when $\alpha = \beta = 1$. Next we will see why the 2^{n-3} branch appears.

Theorem 4.2.4. *For all $\psi \in \mathbb{R}$, $a_n = \psi^n$ is a solution to (4.11) iff*

$$2A_1 + 2A_0 + B_1 + 2B_0 + C_0 + 1 = 0. \tag{4.12}$$

Proof. Let $\psi \in \mathbb{R}$. Assume that $a_n = \psi^n$ is a solution to (4.11). Then

$$\begin{aligned} & \left((\psi^{n+2})^2 (\psi^n)^2 + A_1 \psi^{n+2} (\psi^{n+1})^2 \psi^n + A_0 (\psi^{n+1})^4 \right) (\psi^{n+1})^2 (\psi^{n+3})^2 + \\ & + \left(A_1 (\psi^{n+2})^2 (\psi^n)^2 + B_1 \psi^{n+2} (\psi^{n+1})^2 \psi^n + B_0 (\psi^{n+1})^4 \right) \psi^{n+1} (\psi^{n+2})^2 \psi^{n+3} + \\ & + \left(A_0 (\psi^{n+2})^2 (\psi^n)^2 + B_0 \psi^{n+2} (\psi^{n+1})^2 \psi^n + C_0 (\psi^{n+1})^4 \right) (\psi^{n+2})^4 = 0. \end{aligned}$$

Simplifying, we have

$$\begin{aligned} & (\psi^{4n+4} + A_1 \psi^{4n+4} + A_0 \psi^{4n+4}) \psi^{4n+8} + \\ & + (A_1 \psi^{4n+4} + B_1 \psi^{4n+4} + B_0 \psi^{4n+4}) \psi^{4n+8} + \\ & + (A_0 \psi^{4n+4} + B_0 \psi^{4n+4} + C_0 \psi^{4n+4}) \psi^{4n+8} = 0. \end{aligned}$$

We divide both sides by ψ^{8n+12} and simplify to get

$$2A_1 + 2A_0 + B_1 + 2B_0 + C_0 + 1 = 0.$$

Therefore, if $a_n = \psi^n$, then (4.12) is true. This also gives us the reverse implication: if (4.12) is true, then ψ^n satisfies the recurrence (4.11) for any $\psi \in \mathbb{R}$. \square

Corollary 4.2.5. *For all $\psi \in \mathbb{R}$, $s_n = \psi^n$ is annihilated by the order 3 recurrence (4.9) for the generalized Somos-4 sequence,*

$$s_n^2 s_{n+3}^2 + (\alpha s_{n+1}^3 - (2\alpha + \beta + 1) s_{n+2} s_{n+1} s_n) s_{n+3} + \alpha s_{n+2}^3 s_n + \beta s_{n+2}^2 s_{n+1}^2 = 0.$$

Proof. If $A_1 = A_0 = 0$, $B_1 = -(2\alpha + \beta + 1)$, $B_0 = \alpha$, and $C_0 = \beta$, as in the generalized Somos-4 case, then equation (4.12) in Theorem 4.2.4 is satisfied. \square

Now we understand why the 2^{n-3} branch occurs in the sequence tree for the original Somos-4. Note, however, that the fact that ψ^n is annihilated by (4.9) doesn't guarantee that we will see it as a branch in the sequence tree with initial conditions uniformly equal to 1. Exponential branches will only appear if the terms ψ^j , ψ^{j+1} , and ψ^{j+2} appear consecutively, as a_n , a_{n+1} , and a_{n+2} , on a branch in the sequence tree. We can, however, force them to appear by setting the initial conditions: $a_1 = 1$, $a_2 = \psi$, and $a_3 = \psi^2$.

Together, Theorems 4.2.2 and 4.2.4 give us the following theorem, in the same spirit as Hone and Swart's Theorem 4.2.1.

Theorem 4.2.6. *Given $A_0, A_1, B_0, B_1, C_0 \in \mathbb{Z}$, consider the following octic equation:*

$$\begin{aligned} s^2t^2u^2v^2 + A_0(t^6v^2 + s^2u^6) + A_1(st^4uv^2 + s^2tu^4v) + \\ + B_0(t^5u^2v + st^2u^5) + B_1st^3u^3v + C_0t^4u^4 = 0 \end{aligned} \quad (4.13)$$

The following sequences $\{a_n\}_{n=1}^{\infty}$, where $s = a_n$, $t = a_{n+1}$, $u = a_{n+2}$, and $v = a_{n+3}$, produce infinitely many integer solutions of the Diophantine equation (4.13):

(i) $a_n = \gamma^{\lfloor n^2/4 \rfloor}$ iff γ is a solution to

$$(A_1 + A_0 + 1)\gamma^2 + (A_1 + B_1 + B_0)\gamma + A_0 + B_0 + C_0 = 0.$$

(ii) $a_n = \psi^n$ (for any $\psi \in \mathbb{R}$) iff

$$2A_1 + 2A_0 + B_1 + 2B_0 + C_0 + 1 = 0.$$

4.3 Conclusion

In this chapter we focused on surprising sequences produced by m -recurrences: multivariable polynomials in which one of the variables has degree m , treated from the perspective of recurrences. In general we expect m -recurrences to produce complex numbers, so it was unexpected when we found families that produce rational numbers. We began by considering the sequence of ratios of ratios of a generalized Somos-4 sequence. The natural recurrence for this sequence is not an m -recurrence, however by using a result of Hone and Swart we arrived at a 2-recurrence of order 1. Then we proved that a more general version of this 2-recurrence produces rational numbers, and was not specifically constructed to do so. This new 2-recurrence happened to be the Euler-Chasles correspondence which is known to have connections to many other areas in mathematics.

Next, we made the substitution $f_n = \frac{s_{n+2}s_n}{s_{n+1}^2}$ into the 2-recurrence of order 1 for the ratios of ratios of the generalized Somos-4 sequence to obtain a 2-recurrence of order 3. We observed that this 2-recurrence annihilates the Somos-4 sequence, and also noticed that sequences of exponentials were being produced. This is quite surprising for two reasons: first the known closed form formula for Somos-4 is in terms of the Weierstrass

sigma function, but the closed form of the exponential sequences are of course in terms of an elementary function; secondly, the Somos-4 sequence is known to be asymptotically ϕ^{n^2} , and this order 3 recurrence produces sequences that are asymptotically ψ^n . We then observed that exponential sequences were also being produced by the more general 2-recurrence of order 3 obtained by making an analogous substitution into the Euler-Chasles correspondence.

Within this chapter, and throughout the thesis, the use of computers and programming was invaluable. Without experimentation many, if not all, of the results in Chapters 3 and 4 would not have been conjectured. In addition, Lemmas 3.1.3 and 3.2.1, the piecewise polynomial expressions for the initial terms produced by two similar nonlinear recurrences, were first conjectured and proved by a computer program written for Maple. Finally, and most conspicuously, the usefulness of my algorithm presented in Chapter 2 relies heavily on programming. As you can see in Appendix A.3 the polynomials that are produced, even for small K values, are very large. Making the needed substitutions and expansions is not feasible by hand.

We have now seen that the general topic of nonlinear recurrences provides many interesting phenomena to study. The idea of global asymptotic stability, or convergence independent of initial conditions, is so simple to state, yet quite difficult to prove. Through the use of computers and experimentation I was able to create a very general algorithm to prove global asymptotic stability.

Additionally studying the terms of the sequences produced by nonlinear recurrences can also be quite fascinating. When we expect sequences to be rational and observe integers, as was the case in Chapter 3, or complex and observe rational, as in Chapter 4, there can be many ways to arrive at a proof of this fact. However, I believe that our most valuable tool is experimentation. Though it does not always lead to a proof, it invariably leads to more insight and a better understanding of the problem.

Appendix A

Global Asymptotic Stability

A.1 Summary Table of GAS Results

The equation numbers given in the following table match up with those in [3], however the difference equations themselves may look different. The parameters presented here are to guarantee that the equilibria will be rational functions in the parameters, which is necessary for the algorithm to work.

Eqn #	$x_{n+1} =$	Parameter Values	Findings
2	$\frac{M^2}{x_n}$	$M \in \mathbb{R}$	$\bar{x} = M $ is not LAS
3	$\frac{M^2}{x_{n-1}}$	$M \in \mathbb{R}$	$\bar{x} = M $ is not LAS
5	βx_n	$0 \leq \beta < 1$ $1 \leq \beta$	$\bar{x} = 0$ is GAS $\bar{x} = 0$ is not LAS
9	γx_{n-1}	$0 \leq \gamma < 1$ $1 \leq \gamma$	$\bar{x} = 0$ is GAS $\bar{x} = 0$ is not LAS
17	$\frac{1}{4} \frac{M^2 - 1}{1 + x_n}$	$M - 1 > 0, M + 1 > 0$ $M - 1 < 0, M + 1 < 0$	$\bar{x} = \frac{1}{2}(M - 1)$ is GAS $\bar{x} = -\frac{1}{2}(M + 1)$ is GAS
23	$\frac{\beta x_n}{1 + x_n}$	$0 < \beta \leq 1$ $1 < \beta$	$\bar{x} = 0$ is GAS $\bar{x} = \beta - 1$ is GAS
29	$\frac{x_{n-1}}{A + x_n}$	$0 < A < 1$ $1 < A$	$\bar{x} = 1 - A$ is not LAS $\bar{x} = 0$ is GAS
30	$\frac{x_{n-1}}{A + x_{n-1}}$	$0 < A < 1$ $1 < A$	$\bar{x} = 1 - A$ is GAS $\bar{x} = 0$ is GAS
41	$\alpha + \beta x_n$	$0 \leq \beta < 1$	$\bar{x} = \frac{\alpha}{1 - \beta}$ is GAS
42	$q + \frac{1}{4} \frac{M^2 - q^2}{x_n}$	$M - q < 0, M + q < 0, q > 0$	$\bar{x} = -\frac{1}{2}(M - q)$ is GAS

		$M - q > 0, M + q > 0, q > 0$	$\bar{x} = \frac{1}{2}(M + q)$ is GAS
65	$\frac{1}{4} \frac{M^2 - q^2 + 4x_n}{1 + q + x_n}$	$M - q > 0, M + q > 0, q > -1$	$\bar{x} = \frac{1}{2}(M - q)$ is GAS
		$M - q < 0, M + q < 0, q > -1$	$\bar{x} = -\frac{1}{2}(M + q)$ is GAS
109	$\frac{x_{n-1}}{A + Bx_n + x_{n-1}}$	$1 < A$	$\bar{x} = 0$ is GAS

A.2 GAS Results for Specific Parameter Values

Conjecture A.2.1. Consider the rational difference equation $x_{n+1} = \frac{1}{4} \frac{M^2 - 1}{1 + x_{n-1}}$. For the equilibrium $\bar{x} = \frac{1}{2}(M - 1)$, the following K values are conjectured depending on the value of M .

- If $1 \leq M \leq 3$ then $K = 2$.
- If $3 < M < \infty$ then $K = 4$.

Conjecture A.2.2. Consider the rational difference equation $x_{n+1} = \frac{\beta x_n}{1 + x_{n-1}}$. For the equilibrium $\bar{x} = 0$ and in the range $0 < \beta < 1$, the following K values are conjectured depending on the value of β .

- If $0 \leq \beta \leq \frac{78}{100}$ then $K = 2$.
- If $\frac{79}{100} < \beta < \frac{90}{100}$ then $K = 4$.

Conjecture A.2.3. Consider the rational difference equation $x_{n+1} = \frac{x_n}{Bx_n + x_{n-1}}$. For the equilibrium $\bar{x} = \frac{1}{B+1}$, the following K values are conjectured depending on the value of B .

- If $\frac{11}{10} < B$ then $K = 3$.
- If $\frac{2}{5} < B < \frac{11}{10}$ then $K = 5$.
- If $\frac{1}{5} < B < \frac{2}{5}$ then $K = 6$.

More conjectures along the lines of these, as well as evidence for their truth can be found on my website in WebBooks.

A.3 Polynomials from Section 2.4

$$\begin{aligned}
P_{NE} = & 25x_1^8x_2^4 + 540x_1^8x_2^3 + 4246x_1^8x_2^2 + 14364x_1^8x_2 + 17689x_1^8 + 60x_1^7x_2^5 + 2158x_1^7x_2^4 + \\
& + 28764x_1^7x_2^3 + 180224x_1^7x_2^2 + 534792x_1^7x_2 + 605682x_1^7 + 71x_1^6x_2^6 + 3110x_1^6x_2^5 + \\
& + 57081x_1^6x_2^4 + 553490x_1^6x_2^3 + 2915077x_1^6x_2^2 + 7801588x_1^6x_2 + 8271631x_1^6 + 72x_1^5x_2^7 + \\
& + 3280x_1^5x_2^6 + 64380x_1^5x_2^5 + 730554x_1^5x_2^4 + 5281732x_1^5x_2^3 + 23561650x_1^5x_2^2 + \\
& + 57561520x_1^5x_2 + 57949836x_1^5 + 47x_1^4x_2^8 + 2748x_1^4x_2^7 + 62788x_1^4x_2^6 + 761208x_1^4x_2^5 + \\
& + 5670540x_1^4x_2^4 + 29017604x_1^4x_2^3 + 103895935x_1^4x_2^2 + 228338316x_1^4x_2 + 220764951x_1^4 + \\
& + 12x_1^3x_2^9 + 1130x_1^3x_2^8 + 37294x_1^3x_2^7 + 606604x_1^3x_2^6 + 5507388x_1^3x_2^5 + 29733820x_1^3x_2^4 + \\
& + 103008074x_1^3x_2^3 + 256148328x_1^3x_2^2 + 464566608x_1^3x_2 + 435663522x_1^3 + x_1^2x_2^{10} + \\
& + 178x_1^2x_2^9 + 9489x_1^2x_2^8 + 232730x_1^2x_2^7 + 3086529x_1^2x_2^6 + 23530738x_1^2x_2^5 + \\
& + 103849099x_1^2x_2^4 + 259435104x_1^2x_2^3 + 367065783x_1^2x_2^2 + 378499716x_1^2x_2 + \\
& + 349366689x_1^2 + 8x_1x_2^{10} + 846x_1x_2^9 + 33714x_1x_2^8 + 684036x_1x_2^7 + 7901294x_1x_2^6 + \\
& + 53857984x_1x_2^5 + 212832180x_1x_2^4 + 447797070x_1x_2^3 + 384006582x_1x_2^2 - \\
& - 6980904x_1x_2 + 16x_2^{10} + 1304x_2^9 + 43289x_2^8 + 768660x_2^7 + 8026990x_2^6 + \\
& + 50638188x_2^5 + 188710767x_2^4 + 380507382x_2^3 + 318700575x_2^2
\end{aligned}$$

$$\begin{aligned}
P_{NW} = & 4x_1^8x_2^{10} + 228x_1^8x_2^9 + 5529x_1^8x_2^8 + 74820x_1^8x_2^7 + 621878x_1^8x_2^6 + 3296596x_1^8x_2^5 + \\
& + 11306515x_1^8x_2^4 + 25735114x_1^8x_2^3 + 42709311x_1^8x_2^2 + 57397664x_1^8x_2 + 46388516x_1^8 + \\
& + 20x_1^7x_2^{10} + 1190x_1^7x_2^9 + 29930x_1^7x_2^8 + 417748x_1^7x_2^7 + 3561490x_1^7x_2^6 + \\
& + 19231576x_1^7x_2^5 + 66436356x_1^7x_2^4 + 149132462x_1^7x_2^3 + 237320374x_1^7x_2^2 + \\
& + 308478856x_1^7x_2 + 252636868x_1^7 + 43x_1^6x_2^{10} + 2675x_1^6x_2^9 + 69889x_1^6x_2^8 + \\
& + 1007536x_1^6x_2^7 + 8820526x_1^6x_2^6 + 48545426x_1^6x_2^5 + 168770524x_1^6x_2^4 + \\
& + 371564287x_1^6x_2^3 + 556519171x_1^6x_2^2 + 682931584x_1^6x_2 + 567138751x_1^6 + 52x_1^5x_2^{10} + \\
& + \frac{6771}{2}x_1^5x_2^9 + \frac{183983}{2}x_1^5x_2^8 + 1371361x_1^5x_2^7 + \frac{24682083}{2}x_1^5x_2^6 + 69275600x_1^5x_2^5 + \\
& + 242288688x_1^5x_2^4 + \frac{1041223031}{2}x_1^5x_2^3 + \frac{1433883725}{2}x_1^5x_2^2 + 796874438x_1^5x_2 + \\
& + 671351934x_1^5 + \frac{155}{4}x_1^4x_2^{10} + \frac{10565}{4}x_1^4x_2^9 + \frac{597571}{8}x_1^4x_2^8 + \frac{2304961}{2}x_1^4x_2^7 + \\
& + \frac{42682727}{4}x_1^4x_2^6 + 61146398x_1^4x_2^5 + \frac{1721911435}{8}x_1^4x_2^4 + 449986674x_1^4x_2^3 + \\
& + \frac{4434464165}{8}x_1^4x_2^2 + 516837186x_1^4x_2 + \frac{1766924919}{4}x_1^4 + \frac{73}{4}x_1^3x_2^{10} + \frac{10417}{8}x_1^3x_2^9 + \\
& + \frac{306723}{8}x_1^3x_2^8 + \frac{2450327}{4}x_1^3x_2^7 + \frac{46720239}{8}x_1^3x_2^6 + \frac{68398863}{2}x_1^3x_2^5 + \\
& + \frac{485419901}{4}x_1^3x_2^4 + \frac{1975828589}{8}x_1^3x_2^3 + \frac{2119836981}{8}x_1^3x_2^2 + \frac{353791143}{2}x_1^3x_2 + \\
& + \frac{612436545}{4}x_1^3 + \frac{85}{16}x_1^2x_2^{10} + \frac{6345}{16}x_1^2x_2^9 + \frac{194513}{16}x_1^2x_2^8 + \frac{402403}{2}x_1^2x_2^7 + \\
& + \frac{15810465}{8}x_1^2x_2^6 + \frac{94747555}{8}x_1^2x_2^5 + \frac{339955919}{8}x_1^2x_2^4 + \frac{1355697033}{16}x_1^2x_2^3 + \\
& + \frac{1253946771}{16}x_1^2x_2^2 + 25183305x_1^2x_2 + \frac{349366689}{16}x_1^2 + \frac{7}{8}x_1x_2^{10} + \frac{2185}{32}x_1x_2^9 + \\
& + \frac{69721}{32}x_1x_2^8 + \frac{597651}{16}x_1x_2^7 + \frac{12103333}{32}x_1x_2^6 + \frac{9293423}{4}x_1x_2^5 + \frac{67751361}{8}x_1x_2^4 + \\
& + \frac{537116229}{32}x_1x_2^3 + \frac{445397859}{32}x_1x_2^2 + \frac{872613}{8}x_1x_2 + \frac{1}{16}x_2^{10} + \frac{163}{32}x_2^9 + \frac{43289}{256}x_2^8 + \\
& + \frac{192165}{64}x_2^7 + \frac{4013495}{128}x_2^6 + \frac{12659547}{64}x_2^5 + \frac{188710767}{256}x_2^4 + \frac{190253691}{128}x_2^3 + \\
& + \frac{318700575}{256}x_2^2
\end{aligned}$$

$$\begin{aligned}
P_{SE} = & 2025x_1^8x_2^{10} + 13185x_1^8x_2^9 + \frac{152629}{4}x_1^8x_2^8 + 64679x_1^8x_2^7 + \frac{569321}{8}x_1^8x_2^6 + \\
& + \frac{425171}{8}x_1^8x_2^5 + \frac{873625}{32}x_1^8x_2^4 + \frac{152503}{16}x_1^8x_2^3 + \frac{554437}{256}x_1^8x_2^2 + \frac{74081}{256}x_1^8x_2 + \\
& + \frac{17689}{1024}x_1^8 + 59490x_1^7x_2^{10} + 392258x_1^7x_2^9 + \frac{2301061}{2}x_1^7x_2^8 + 1978050x_1^7x_2^7 + \\
& + \frac{8835689}{4}x_1^7x_2^6 + \frac{6700919}{4}x_1^7x_2^5 + \frac{13989249}{16}x_1^7x_2^4 + \frac{2482117}{8}x_1^7x_2^3 + \frac{9175165}{128}x_1^7x_2^2 + \\
& + \frac{1246809}{128}x_1^7x_2 + \frac{302841}{512}x_1^7 + 719163x_1^6x_2^{10} + 4787551x_1^6x_2^9 + \frac{56742867}{4}x_1^6x_2^8 + \\
& + 24652235x_1^6x_2^7 + \frac{222746539}{8}x_1^6x_2^6 + \frac{170955505}{8}x_1^6x_2^5 + \frac{361391103}{32}x_1^6x_2^4 + \\
& + \frac{64966991}{16}x_1^6x_2^3 + \frac{243455119}{256}x_1^6x_2^2 + \frac{33556567}{256}x_1^6x_2 + \frac{8271631}{1024}x_1^6 + \\
& + 4648948x_1^5x_2^{10} + 31115612x_1^5x_2^9 + 92712559x_1^5x_2^8 + 162071968x_1^5x_2^7 + \\
& + \frac{368447705}{2}x_1^5x_2^6 + \frac{284775455}{2}x_1^5x_2^5 + \frac{606743359}{8}x_1^5x_2^4 + \frac{55019841}{2}x_1^5x_2^3 + \\
& + \frac{416470465}{64}x_1^5x_2^2 + \frac{58046915}{64}x_1^5x_2 + \frac{14487459}{256}x_1^5 + 17579931x_1^4x_2^{10} + \\
& + 117432519x_1^4x_2^9 + \frac{1395861723}{4}x_1^4x_2^8 + 608178197x_1^4x_2^7 + \frac{5514235755}{8}x_1^4x_2^6 + \\
& + \frac{4252314293}{8}x_1^4x_2^5 + \frac{9050945079}{32}x_1^4x_2^4 + \frac{1642995557}{16}x_1^4x_2^3 + \frac{6239916847}{256}x_1^4x_2^2 + \\
& + \frac{875486439}{256}x_1^4x_2 + \frac{220764951}{1024}x_1^4 + 40895890x_1^3x_2^{10} + 268768706x_1^3x_2^9 + \\
& + \frac{1565987989}{2}x_1^3x_2^8 + 1333796560x_1^3x_2^7 + \frac{5897895725}{4}x_1^3x_2^6 + \frac{4433312709}{4}x_1^3x_2^5 + \\
& + \frac{9209512797}{16}x_1^3x_2^4 + \frac{818859649}{4}x_1^3x_2^3 + \frac{6133626429}{128}x_1^3x_2^2 + \frac{856875501}{128}x_1^3x_2 + \\
& + \frac{217831761}{512}x_1^3 + 63839013x_1^2x_2^{10} + 404278497x_1^2x_2^9 + \frac{4488027277}{4}x_1^2x_2^8 + \\
& + 1797399271x_1^2x_2^7 + \frac{14739034957}{8}x_1^2x_2^6 + \frac{10128161251}{8}x_1^2x_2^5 + \frac{19011819929}{32}x_1^2x_2^4 + \\
& + \frac{3042830919}{16}x_1^2x_2^3 + \frac{10376769249}{256}x_1^2x_2^2 + \frac{1368333729}{256}x_1^2x_2 + \frac{349366689}{1024}x_1^2 + \\
& + 75803000x_1x_2^{10} + 464366336x_1x_2^9 + 1226670172x_1x_2^8 + 1826750502x_1x_2^7 + \\
& + 1679123734x_1x_2^6 + \frac{1953449263}{2}x_1x_2^5 + \frac{1406888001}{4}x_1x_2^4 + \frac{575528697}{8}x_1x_2^3 + \\
& + \frac{207710325}{32}x_1x_2^2 + \frac{872613}{32}x_1x_2 + 55463100x_2^{10} + 342684440x_2^9 + 913467212x_2^8 + \\
& + 1373326010x_2^7 + \frac{2549536925}{2}x_2^6 + \frac{1497468765}{2}x_2^5 + \frac{1087838955}{4}x_2^4 + \\
& + \frac{447147459}{8}x_2^3 + \frac{318700575}{64}x_2^2
\end{aligned}$$

$$\begin{aligned}
P_{SW} = & 3488704x_1^8x_2^{10} + 23706428x_1^8x_2^9 + 71233417x_1^8x_2^8 + 124708250x_1^8x_2^7 + \\
& + 141003789x_1^8x_2^6 + 107781737x_1^8x_2^5 + \frac{452745187}{8}x_1^8x_2^4 + \frac{162175447}{8}x_1^8x_2^3 + \\
& + \frac{76572657}{16}x_1^8x_2^2 + \frac{43636229}{64}x_1^8x_2 + \frac{11597129}{256}x_1^8 + 21009116x_1^7x_2^{10} + \\
& + 141731236x_1^7x_2^9 + 422322457x_1^7x_2^8 + 732306258x_1^7x_2^7 + \frac{1638161821}{2}x_1^7x_2^6 + \\
& + 618619029x_1^7x_2^5 + \frac{2565092523}{8}x_1^7x_2^4 + \frac{906696175}{8}x_1^7x_2^3 + \frac{1691330287}{64}x_1^7x_2^2 + \\
& + \frac{238676371}{64}x_1^7x_2 + \frac{63159217}{256}x_1^7 + 53827795x_1^6x_2^{10} + 359954471x_1^6x_2^9 + \\
& + \frac{4245985867}{4}x_1^6x_2^8 + 1818444920x_1^6x_2^7 + \frac{16044958663}{8}x_1^6x_2^6 + \frac{11926554211}{8}x_1^6x_2^5 + \\
& + \frac{24294753531}{32}x_1^6x_2^4 + \frac{2107604703}{8}x_1^6x_2^3 + \frac{15454551967}{256}x_1^6x_2^2 + \frac{2152762171}{256}x_1^6x_2 + \\
& + \frac{567138751}{1024}x_1^6 + 76440336x_1^5x_2^{10} + 505681254x_1^5x_2^9 + \frac{2943815873}{2}x_1^5x_2^8 + \\
& + \frac{4964759983}{2}x_1^5x_2^7 + \frac{10749765175}{4}x_1^5x_2^6 + \frac{15635866609}{8}x_1^5x_2^5 + 970763489x_1^5x_2^4 + \\
& + \frac{10491130005}{32}x_1^5x_2^3 + \frac{4683716149}{64}x_1^5x_2^2 + \frac{159992827}{16}x_1^5x_2 + \frac{335675967}{512}x_1^5 + \\
& + \frac{262997769}{4}x_1^4x_2^{10} + \frac{1716787291}{4}x_1^4x_2^9 + \frac{19659889067}{16}x_1^4x_2^8 + \frac{8120807643}{4}x_1^4x_2^7 + \\
& + \frac{68574814865}{32}x_1^4x_2^6 + \frac{48346786897}{32}x_1^4x_2^5 + \frac{92523956871}{128}x_1^4x_2^4 + \frac{14966716723}{64}x_1^4x_2^3 + \\
& + \frac{51168272293}{1024}x_1^4x_2^2 + \frac{6767275851}{1024}x_1^4x_2 + \frac{1766924919}{4096}x_1^4 + \frac{140897099}{4}x_1^3x_2^{10} + \\
& + \frac{904934601}{4}x_1^3x_2^9 + \frac{10149114177}{16}x_1^3x_2^8 + \frac{8162486375}{8}x_1^3x_2^7 + \frac{33285150493}{32}x_1^3x_2^6 + \\
& + \frac{1401534327}{2}x_1^3x_2^5 + \frac{40446119631}{128}x_1^3x_2^4 + \frac{12140134537}{128}x_1^3x_2^3 + \frac{19062837339}{1024}x_1^3x_2^2 + \\
& + \frac{2354600439}{1024}x_1^3x_2 + \frac{612436545}{4096}x_1^3 + \frac{186770213}{16}x_1^2x_2^{10} + \frac{1177787401}{16}x_1^2x_2^9 + \\
& + \frac{12891726805}{64}x_1^2x_2^8 + \frac{627131823}{2}x_1^2x_2^7 + \frac{39110604305}{128}x_1^2x_2^6 + \frac{24708996989}{128}x_1^2x_2^5 + \\
& + \frac{40537937381}{512}x_1^2x_2^4 + \frac{2637097233}{128}x_1^2x_2^3 + \frac{13484496249}{4096}x_1^2x_2^2 + \frac{1343900565}{4096}x_1^2x_2 + \\
& + \frac{349366689}{16384}x_1^2 + \frac{18256175}{8}x_1x_2^{10} + \frac{28324107}{2}x_1x_2^9 + \frac{606799669}{16}x_1x_2^8 + \\
& + \frac{1833276769}{32}x_1x_2^7 + \frac{854987529}{16}x_1x_2^6 + \frac{4036425797}{128}x_1x_2^5 + \frac{2944467819}{256}x_1x_2^4 + \\
& + \frac{1213061139}{512}x_1x_2^3 + \frac{429690825}{2048}x_1x_2^2 - \frac{872613}{2048}x_1x_2 + \frac{13865775}{64}x_2^{10} + \\
& + \frac{42835555}{32}x_2^9 + \frac{228366803}{64}x_2^8 + \frac{686663005}{128}x_2^7 + \frac{2549536925}{512}x_2^6 + \frac{1497468765}{512}x_2^5 + \\
& + \frac{1087838955}{1024}x_2^4 + \frac{447147459}{2048}x_2^3 + \frac{318700575}{16384}x_2^2
\end{aligned}$$

References

- [1] A. Berenstein, S. Fomin, and A. Zelevinsky, Cluster algebras. III. Upper bounds and double Bruhat cells, *Duke Mathematical Journal*, **126**(1), (2005), 1–52.
- [2] V. M. Buchstaber and A. P. Veselov, Integrable correspondences and algebraic representations of multivalued groups, *International Mathematics Research Notes*, (8), (1996), 381–400.
- [3] E. Camouzis and G. Ladas, *Dynamics of Third Order Rational Difference Equations*, Chapman and Hall/CRC press (2008).
- [4] D. Carroll, Gabriel; Speyer, The cube recurrence, *Electronic Journal of Combinatorics*, **11**.
- [5] H. A. El-Metwally, E. A. Grove, and G. Ladas, A global convergence result with applications to periodic solutions, *Journal of Mathematical Analysis and Applications*, **245**, (2000), 161–170.
- [6] H. A. El-Metwally, E. A. Grove, G. Ladas, and H. D. Voulov, On the global attractivity and the periodic character of some difference equations, *Journal of Difference Equations and Applications*, **7**, (2001), 837–850.
- [7] S. Elayadi, *Introduction to Difference Equations*, Springer (2000).
- [8] G. Everest, A. van der Poorten, I. Shparlinski, and T. Ward, *Recurrence Sequences*, vol. 104 of *Mathematical Surveys and Monographs*, American Mathematical Society (2003).
- [9] S. Fomin, Total positivity and cluster algebras, in *Proceedings of the International Congress of Mathematicians*, vol. 2, Hyderabad, India, (125–145).
- [10] S. Fomin and A. Zelevinsky, Cluster algebras. I. Foundations, *Journal of the American Mathematical Society*, **15**(2), (2002), 497–529.
- [11] S. Fomin and A. Zelevinsky, The laurent phenomenon, *Advances in Applied Mathematics*, **28**, (2002), 119–144.
- [12] S. Fomin and A. Zelevinsky, Cluster algebras. II. Finite type classification, *Inventiones Mathematicae*, **154**(1), (2003), 63–121.
- [13] S. Fomin and A. Zelevinsky, Cluster algebras. IV. Coefficients, *Compositio Mathematica*, **143**(1), (2007), 112–164.
- [14] D. Gale, Mathematical entertainments: The strange and surprising saga of the somos sequences, *Mathematical Intelligencer*, **13**, (1991), 40–42.
- [15] J. Hale and H. Kocak, *Dynamics & Bifurcations*, Springer-Verlag (1991).

- [16] M. L. J. Hatus and T. S. Bolis, Solution to problem E2721, *American Mathematical Monthly*, **86**, (1979), 865–866.
- [17] P. Heideman and E. Hogan, A new family of somos-like recurrences, *Electronic Journal of Combinatorics*, **15**(1).
- [18] A. N. W. Hone, Elliptic curves and quadratic recurrence sequences, *Bulletin of the London Mathematical Society*, **27**, (2005), 161–171.
- [19] A. N. W. Hone, Sigma function solution of the initial value problem for Somos 5 sequences, *Transactions of the American Mathematical Society*, **359**(10), (2007), 5019–5034.
- [20] A. N. W. Hone and C. Swart, Integrality and the laurent phenomenon for somos 4 and somos 5 sequences, *Mathematical Proceedings of the Cambridge Philosophical Society*, **145**, (2008), 65–85.
- [21] H. Hong and D. Jakuš, Testing positiveness of polynomials, *Journal of Automatic Reasoning*, **21**(1), (1998), 23–38.
- [22] V. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers (1993).
- [23] N. Kruse and T. Nesemann, Global asymptotic stability in some discrete dynamical systems, *Journal of Mathematical Analysis and Applications*, **235**, (1999), 151–158.
- [24] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations*, Chapman and Hall/CRC press (2001).
- [25] T. Lam, *Introduction to Quadratic Forms over Fields*, vol. 67 of *Graduate Studies in Mathematics*, American Mathematical Society (2004).
- [26] J. W. Layman, The hankel transform and some of its properties, *Journal of Integer Sequences*, **4**.
- [27] J. Malouf, An integer sequence from a rational recursion, *Discrete Mathematics*, **110**, (1992), 257–261.
- [28] M. Martelli, *Introduction to Discrete Dynamical Systems*, John Wiley & Sons (1999).
- [29] OEIS Foundation Inc., The on-line encyclopedia of integer sequences (2011).
URL <http://oeis.org>
- [30] O. T. O’Meara, *Introduction to Quadratic Forms*, Springer-Verlag (1971).
- [31] M. Somos, Problem 1470, *Cruz Mathematicorum*, **15**, (1989), 208.
- [32] M. Somos, Brief history of the somos sequence problem (October 1994).
URL <http://faculty.uml.edu/jpropp/somos/history.txt>
- [33] D. Speyer, Perfect matchings and the octahedron recurrence, *Journal of Algebraic Combinatorics*, **25**(3), (2007), 309–348.

- [34] R. Stanley, *Enumerative Combinatorics, Volume 1*, Cambridge University Press (2002).
- [35] R. Stanley, Recent developments in algebraic combinatorics, *Israel Journal of Mathematics*, **143**, (2004), 317–339.
- [36] C. Swart, *Elliptic Curves and Related Sequences*, Ph.D. thesis, University of London (2003).
- [37] A. P. Veselov, Growth and integrability in the dynamics of mappings, *Communications in Mathematical Physics*, **145**, (1992), 181–193.

Curriculum Vitae

Emilie Ann Hogan

9/2006 – 5/2011 Ph. D. in Mathematics, Rutgers University

9/2002 – 5/2006 B. Sc. in Mathematics, University of Wisconsin – Madison

1/2011 – 5/2011 Teaching assistant, Mathematics Department, Rutgers University

1/2008 – 12/2010 DHS Graduate Fellow, Rutgers University

9/2006 – 12/2007 Teaching assistant, Mathematics Department, Rutgers University

6/2006 – 8/2006 Intern, Wave Technologies

2010 C. Joslyn, E. Hogan. Order Metrics for Semantic Knowledge Systems. In E.S. Corchado Rodriguez et al. (Eds.) *Hybrid Artificial Intelligence Systems 2010, Part II* (399-409). Lecture Notes in Artificial Intelligence 6077, 2010.

2009 C. Joslyn, J. Burke, T. Critchlow, N. Hengartner, E. Hogan. View Discovery in OLAP Databases Through Statistical Combinatorial Optimization. In Marianne Winslett (Ed.) *Scientific and Statistical Database Management* (37-55). Lecture Notes in Computer Science, Volume 5566/2009.

2008 P. Heideman, E. Hogan. A New Family of Somos-Like Recurrences, *Electronic Journal of Combinatorics*, 15(1), 2008.