Computational methods in permutation patterns

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Reduction

We will consider permutations $\pi = \pi_1 \dots \pi_n \in \mathcal{S}_n$ in one-line notation.

Definition

The *reduction* of a sequence of distinct positive integers $s_1 s_2 \dots s_k$, denoted by $\text{red}(s_1 \dots s_k)$, is the length k permutation obtained by relabeling the i-th smallest term by i.

Example

$\text{red}(63915) = 42513$

Classical pattern occurrences

Definition

Given a (permutation) pattern $\tau = \tau_1 \ldots \tau_k$, we say that permutation $\pi = \pi_1 \ldots \pi_n$ *contains* the pattern τ if there exists $1 \leq i_1 < \ldots < i_k \leq n$ such that $\mathrm{red}(\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}) = \tau$.

Example

If pattern $\tau = 123$,

- $\pi = 54321$ has zero occurrences of τ,
- $\pi = 42135$ has two occurrences of τ.

Background

Permutations patterns gained interest after some results in sorting.

Theorem (Knuth, 1968)

A permutation is stack-sortable if and only if it avoids the pattern 231.

This led to interest in enumerative questions.

Definition

Given a pattern τ, define

$$s_n(\tau) := \# \text{ of } \pi \in \mathcal{S}_n \text{ that avoid } \tau.$$

What can we say about $s_n(\tau)$?

Some previous results

Length 3 patterns (Knuth, 1968):

$$s_n(123) = s_n(132) = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Length 4 patterns:

- Closed form for $s_n(1234)$ known. (Gessel, 1990)
- Closed form for $s_n(1342)$ known. (Bóna, 1997)
- $s_n(1324) =???$

Conjecture (Zeilberger, 2005)

"Not even God knows $s_{1000}(1324)$."

Talk outline

We will consider two variations:

1. Enumerating permutations with exactly r copies of a (classical) pattern.
    - Functional equations approach
    - Computationally extending existing techniques
2. Enumerating permutations avoiding consecutive patterns

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r copies of a pattern

Definition

Given a pattern τ and $r \geq 0$, define

$$s_n(\tau, r) := \# \text{ of } \pi \in \mathcal{S}_n \text{ with exactly } r \text{ occurrences of } \tau.$$

Most work on $r > 0$ focuses on length 3 patterns:

- G.F. for $s_n(132, r)$ studied by Bóna, Mansour and Vainshtein, Fulmek, and others.
- G.F. for $s_n(123, r)$ studied by Noonan and Zeilberger, Fulmek, Callan, and others.

GOAL: for fixed pattern τ and fixed r, compute $s_n(\tau, r)$ “quickly”.

We will assume $\tau = 123$ (equiv. abc). (joint with Zeilberger)

Additional definitions

Definition

For variables $t, x_1, \dots, x_n$, define

$$weight(\pi) := t^{\#\text{ of } abc \text{ in } \pi} \cdot \prod_{i=1}^{n} x_i^{\#\text{ of } ab \text{ in } \pi \text{ s.t. } a=i}$$

$$P_n(t; x_1, \dots, x_n) := \sum_{\pi \in \mathcal{S}_n} weight(\pi)$$

Example

$weight(2134) = t^2 x_1^2 x_2^2 x_3$

Observe: coeff. of t^r in $P_n(t; 1, \dots, 1) = s_n(123, r)$.

Functional equations

Noonan-Zeilberger Functional Equation (NZFE)

$$P_n(t; x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^{n-i} P_{n-1}(t; x_1, \ldots, x_{i-1}, tx_{i+1}, \ldots, tx_n)$$

We can use this functional equation to compute $P_n(t; 1, \ldots, 1)$.

Maple implementation

Can apply other computational methods to quickly find coeff. of t^r in $P_n(t; 1, \ldots, 1)$ (i.e., $s_n(123, r)$).
(*in polynomial-time!*)

Everything has been implemented in Maple:

Example

- For $r = 0$, the first 10 terms of $s_n(123, r)$ are: $1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796$
- For $r = 1$, the first 10 terms of $s_n(123, r)$ are: $0, 0, 1, 6, 27, 110, 429, 1638, 6188, 23256$
- For $r = 6$, the values of $s_n(123, r)$ for $15 \leq n \leq 20$ are: $327200581, 1501719377, 6773007550, 30100185693, 132099138291, 573518305776$

Some extensions

The enumeration approach can be extended to:

- Any increasing pattern $12\ldots k$. (joint with Zeilberger)
  (For example, $s_{60}(1234, 1)$ is:
  234261080605837210966025910570764305425250198302448)
- Patterns 132, 1243, and more generally $12\ldots(k-2)k(k-1)$.
  (For example, $s_{60}(1243, 1)$ is:
  286623815577790281658916215981275905173953218878 7)
- Certain cases of multiple patterns
- Refining by inversions

Additional extensions

This approach can be generalized to handle other patterns by considering more complicated catalytic variables $x_{i,j}$'s.

Some additional patterns that can be handled with this approach:

- Patterns 231, 2341, and more generally $23\ldots k1$.
- The pattern 1324

Set-up for 1324

We consider the catalytic variables:

- $x_{i,j}$ $(1 \leq i \leq j \leq n)$
- $y_{i,j}$ $(1 \leq j \leq i \leq n)$.

Variables $x_{i,j}$ will be written as a matrix of variables:

$$X_n := \begin{bmatrix} x_{1,1} & & \cdots & & x_{1,n} \\ & \ddots & & & \\ \vdots & & x_{i,i} & & \vdots \\ & & & \ddots & \\ x_{n,1} & & \cdots & & x_{n,n} \end{bmatrix}$$

(similarly for variables $y_{i,j}$ and matrix Y_n)

Functional equation for 1324

We define a polynomial $P_n(t; X_n, Y_n)$ so that coeff. of t^r in $P_n(t; \mathbf{1}, \mathbf{1})$ is exactly $s_n(1324, r)$.

We can then derive the functional equation:

$$P_n(t; X_n, Y_n) = \sum_{i=1}^{n} x_{i,i}^{n-i} x_{i,i+1}^{n-i-1} \dots x_{i,n-1}^{1} \cdot P_{n-1}(t;\ R_2(X_n, Y_n, i),\ R_1(Y_n, i))$$

(with some matrix operators R_1 and R_2).

Improvements to 1324

We can also specialize the functional equation for the $r = 0$ case.

This allows us to compute the first 23 terms.
For example, $s_{23}(1324) = 94944352095728825$.

Easy to refine by the number of inversions.

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Generating function

Definition

Given a pattern τ and fixed $r \geq 0$, define

$$F_\tau^r(x) := \sum_{n=0}^{\infty} s_n(\tau, r)x^n.$$

Recall that Dyck paths are counted by the Catalan numbers.

Generating function: $C(x) = \frac{1-\sqrt{1-4x}}{2x}$.

Fulmek's approach

We will consider the pattern 312.

Fulmek gave an approach to compute $F_{312}^{r}(x)$ for $r = 1, 2$.

GENERAL IDEA:

- Map permutation into a "generalized Dyck path" (a Dyck path where down-jumps are allowed).
- Count the relevant paths.

Mapping is injective, and the down-jumps will mark the occurrences of 312.

GOAL: study Fulmek's approach and extend it to larger r.

Finding $F^1_{312}(x)$

The permutation 312 has the corresponding path:

The paths corresponding to permutations with 1 copy of 312 will contain this subpath (and no other down-jumps).

Find the generating function counting such paths.

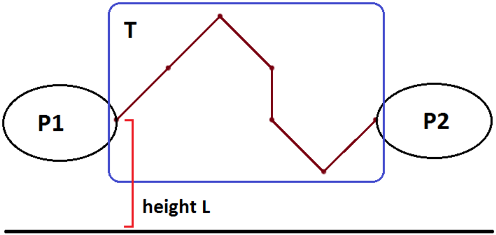
Finding $F^1_{312}(x)$

The permutation 312 has the corresponding path:

The paths corresponding to permutations with 1 copy of 312 will contain this subpath (and no other down-jumps).

Find the generating function counting such paths.

Finding $F^1_{312}(x)$ (cont'd)



"weight" of up/down-steps $= x^{1/2}$; "weight" of down-jumps $= 1$

"weight" of T $= x^{5/2}$
"weight" of all P1 paths = "weight" of all P2 paths $= x^{L/2}C^{L+1}$

$$F^1_{312}(x) = \frac{1}{x^{1/2}} \sum_{L=1}^{\infty} \text{weight}(P1) \cdot \text{weight}(T) \cdot \text{weight}(P2) = \frac{C^4x^3}{1 - C^2x}$$

Finding $F_{312}^{2}(x)$

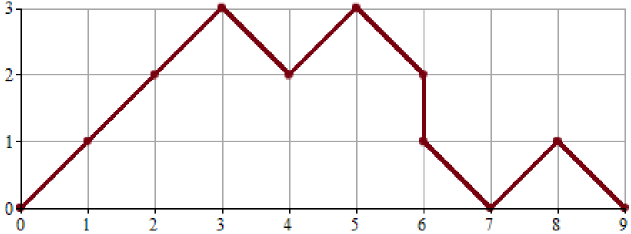
"ROUGH IDEA":

- Find "base permutations" for two occurrences of 312: 3412, 4132, 4213, 4312, 31524, 312645, 316452, 423615
- Find the generating function for each one.
- Add the generating functions together to get $F_{312}^{2}(x)$.

By again considering subpaths in generalized Dyck paths, we can reduce the number of cases that need to be handled.

Finding $F^2_{312}(x)$: 3412 case

The base permutation 3412 has the corresponding path:



And the corresponding generating function is:

$$\frac{C^4x^4}{1 - C^2x}$$

Finding $F^2_{312}(x)$: 3412 case

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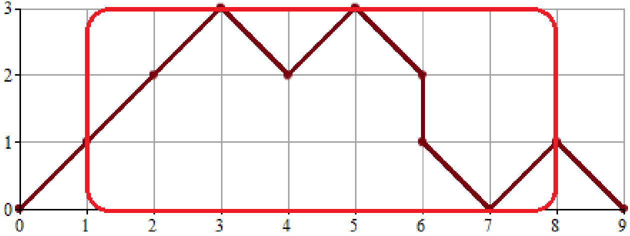
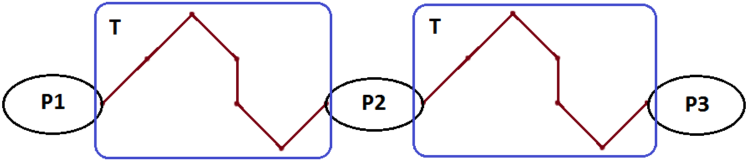


And the corresponding generating function is:

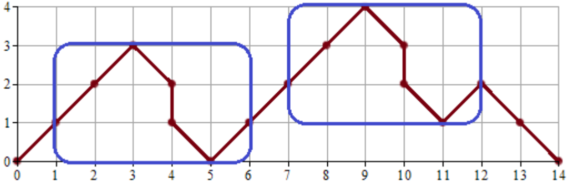
$$\frac{C^4x^4}{1-C^2x}$$

Finding $F^2_{312}(x)$: two 312's

For two disjoint 312 patterns, we have the path structure:



For example, the path for 316452 is:



Finding $F^2_{312}(x)$ and more

Combining all the generating functions, we can find $F^2_{312}(x)$.
NOTE: Fulmek did this in his paper but used various observations to handle some cases.

We were able to make this approach more systematic and automate it in Maple.

We can compute $F^3_{312}(x)$ and $F^4_{312}(x)$ through this same approach.
NOTE: These were also discovered by Mansour and Vainshtein through a different approach.

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Consecutive patterns

Definition

Given a pattern $\sigma = \sigma_1 \cdots \sigma_k$, we say that permutation $\pi = \pi_1 \cdots \pi_n$ *contains* the pattern σ *consecutively* if there exists an i such that $\text{red}(\pi_i \cdots \pi_{i+k-1}) = \sigma$.

Example

If $\sigma = 1243$,

- $\pi = 123654$ contains σ consecutively since $\text{red}(2365) = 1243$.
- $\pi = 12453$ avoids σ.

Consecutive avoidance

Definition

Given a pattern σ, define

$$\alpha_\sigma(n) = \# \text{ of } \pi \in \mathcal{S}_n \text{ such that } \pi \text{ avoids } \sigma \text{ consecutively.}$$

Definition

Define the EGF of $\alpha(n)$ as

$$A_\sigma(z) = \sum_{n=0}^{\infty} \alpha(n) \frac{z^n}{n!}.$$

Background

There are more patterns to consider in consecutive case.

Length 3 patterns: 123 and 132.
(these were equivalent in classical pattern avoidance)

Length 4 patterns: 1234, 2413, 2143, 1324, 1423, 1342, and 1243.
(only 3 patterns in classical pattern avoidance)

Many current "solutions" for the EGF are given as differential equations that $A(z)$ satisfies or as complicated recurrences.

Cluster method

We develop an automated approach based off of an extension of the cluster method.

For any given pattern σ, we can derive a corresponding recurrence:

$$\alpha(n) = n\alpha(n-1) + \sum_{k=1}^{n} \binom{n}{k} C(k)\alpha(n-k)$$

where $C(k)$ is a weighted sum of length k "clusters" of σ.

Computing the $C(k)$ terms will determine $\alpha(n)$.

Example: cluster recurrence for 132

If pattern $\sigma = 132$:

$$C(k) = \sum_{1 \le x_1 < \cdots < x_3 \le k} C(k; [x_1, \ldots, x_3]).$$

For $k < 3$:

$$C(k; [x_1, x_2, x_3]) = 0$$

For $k = 3$:

$$C(k; [x_1, x_2, x_3]) = -1$$

For $k > 3$:

$$C(k; [x_1, x_2, x_3]) = \sum_{\substack{1 \le y_1 < y_2 < y_3 \le k-2 \\ y_2 = x_1}} -C(k-2; [y_1, y_2, y_3])$$

Automated enumeration

We can "teach" a computer to compute $\alpha(n)$ for any given pattern and a specific value of n with the steps:

1. Derive recurrence for $C(k;[x_1,\ldots,x_m])$.
2. Compute $C(k)$ terms.
3. Compute $\alpha(n)$ using recurrence on $\alpha(n)$ and $C(k)$.

(NOTE: the $C(k;[x_1,\ldots,x_m])$ recurrence can be converted to a functional equation)

Example

For the pattern $\sigma = 2143$, we can easily compute $\alpha(45)$:

1825442282343560807118159376065311731253383988747230660

Consecutive Wilf-equivalence

The previous approach provides a rigorous result:

Theorem (Khoroshkin and Shapiro; N.)

Given patterns σ and τ of the same length, if they have the "same self-overlaps", then $A_\sigma(z) = A_\tau(z)$ (consecutively Wilf-equivalent).

The theorem along with the previous algorithm allows us to classify all c-Wilf-equivalence classes up to length 6 patterns*.

Thank you

Thank you!