Solutions to Dr. Z.'s Math 354 REAL Quiz #8

1. (3 points) Describe in as much detail as possible the solution(s) to the dual problem if you know that

"An optimal solution is $[1, 0, 2, 0, 3, 0, 5, 0, 0]^T$ with objective function value 1000 ."

Sol. to 1: We know right away that the optimal objective function value of the dual problem is 1000. We also know that the first, third, fifth, and seventh constraints of the dual problem are **equalities** (rather than inequalities), when you plug-in the optimal solution of the dual problem. This is because the first, third, fifth, and seventh values of the optimal solution for the primal problem are **non-zero**.

2. (5 points) Suppose that $x_1 = 2, x_2 = 0, x_3 = 4$ is an optimal solution to the linear programming problem

Maximize $z = 4x_1 + 2x_2 + 3x_3$

subject to

$$2x_1 + 3x_2 + x_3 \leq 12$$

$$x_1 + 4x_2 + 2x_3 \leq 10$$

$$3x_1 + x_2 + x_3 \leq 10$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0 \quad , \quad x_3 \geq 0 \quad .$$

Using the principle of complementary slackness and the duality theorem, find an optimal solution to the dual problem. What value of the objective function of the dual problem have at this optimal solution?

Sol. 2: The second part is easier (and does not require setting up the dual problem). Since the **optimal value** of the primal problem is

$$z = 4 \cdot 2 + 2 \cdot 0 + 3 \cdot 4 = 8 + 12 = 20 \quad ,$$

it follows by the **Duality Theorem** that the optimal value of the dual problem is also 20.

Regarding the first part, we have two kinds of clues. Since x_1 and x_3 are **non-zero** we know that the first and third constraints of the dual problem are **equalities**.

To get the second kind of clue(s), we plug in the optimal solution into the constraints

• Regarding the first constraint

$$2 \cdot 2 + 3 \cdot 0 + 4 \leq 12 \quad i.e. \quad 8 \leq 12 \quad has \quad slack \quad ,$$

hence we know right away that $w_1 = 0$

• Regarding the second constraint

$$2 + 4 \cdot 0 + 2 \cdot 4 \leq 10 \quad i.e. \quad 10 \leq 10 \quad is \quad tight \quad ,$$

hence we can't conclude anything about w_2.

• Regarding the third constraint

$$3 \cdot 2 + 0 + 4 \leq 10 \quad i.e. \quad 10 \leq 10 \quad is \quad tight \quad ,$$

hence we we can't conclude anything about w_3.

Now is the time to set-up the dual problem

Minimize $z' = 12w_1 + 10w_2 + 10w_3$ subject to

$$2w_1 + w_2 + 3w_3 \geq 4 \quad ,$$

$$3w_1 + 4w_2 + w_3 \geq 2 \quad ,$$

$$w_1 + 2w_2 + w_3 \geq 3 \quad ,$$

$$w_1 \geq 0 \quad , \quad w_2 \geq 0 \quad , \quad w_3 \geq 0 \quad .$$

Replacing the first and third constraints by **equalities** and plugging in $w_1 = 0$ this becomes

$$2 \cdot 0 + w_2 + 3w_3 = 4 \quad ,$$

$$3 \cdot 0 + 4w_2 + w_3 \geq 2 \quad ,$$

$$0 + 2w_2 + w_3 = 3 \quad ,$$

We get the system of two equations $\{w_2 + 3w_3 = 4\, ,\, 2w_2 + w_3 = 3\}$ in the two unknowns $\{w_2, w_3\}$, whose solution is $w_2 = 1$, $w_3 = 1$. Combining with $w_1 = 0$ we get

Ans. to 1: An optimal solution to the dual problem is $(w_1, w_2, w_3) = (0, 1, 1)$ and the the optimal value of the dual problem is also 20.

Comment: It is always good to check that what we know from the duality theorem agrees. Indeed plugging-in the optimal solution of the dual problem $w_1 = 0$ $w_2 = 1$, $w_3 = 1$ into its objective function yields $z' = 12w_1 + 10w_2 + 10w_3$, that gives $z' = 12 \cdot 0 + 10 \cdot 1 + 10 \cdot 1 = 20$, yea!