Solutions to Dr. Z.'s Math 354 REAL Quiz #7

1. (8 pts.) Find the dual of the given linear programming problem.

Minimize $5\,x_1\,+\,2\,x_2\,+\,6\,x_3$

subject to

$$4x_1 + 2x_2 + x_3 \geq 12 \quad , \quad 3x_1 + 2x_2 + 3x_3 \leq 6 \quad ,$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0 \quad , \quad x_3 \geq 0 \quad .$$

Sol. to 1 We first convert do **pre-processing** by converting to **standard form**.

Maximize $z = -5\,x_1\,-\,2\,x_2\,-\,6\,x_3$

subject to

$$-4x_1 - 2x_2 - x_3 \leq -12 \quad , \quad 3x_1 + 2x_2 + 3x_3 \leq 6 \quad ,$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0 \quad , \quad x_3 \geq 0 \quad .$$

In matrix notation this is

Maximize $\mathbf{c^T x}$ subject to

$$A\mathbf{x} \leq \mathbf{b} \quad , \quad \mathbf{x} \geq \mathbf{0} \quad ,$$

where

$$\mathbf{c} = \begin{bmatrix} -5 \\ -2 \\ -6 \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} -12 \\ 6 \end{bmatrix} \quad , \quad A = \begin{bmatrix} -4 & -2 & -1 \\ 3 & 2 & 3 \end{bmatrix}$$

The **dual** problem is

Minimize $\mathbf{b^T w}$ subject to

$$A^T\mathbf{w} \geq \mathbf{c} \quad , \quad \mathbf{w} \geq \mathbf{0} \quad ,$$

where $\mathbf{c}$ and $\mathbf{b}$ are as above, and the transpose of A, A^T is

$$A^T = \begin{bmatrix} -4 & 3 \\ -2 & 2 \\ -1 & 3 \end{bmatrix} \quad .$$

This spells out to

Minimize $z' = -12w_1 + 6w_2$ subject to the constraints

$$-4w_1 + 3w_2 \geq -5 \quad ,$$

$$-2w_1 + 2w_2 \geq -2 \quad ,$$

$$-w_1 + 3w_2 \geq -6 \quad ,$$

$$w_1 \geq 0 \quad , \quad w_2 \geq 0 \quad .$$

This is an acceptable answer.

But it can (optionally) be made nicer by **post-processing**.

Maximize $z' = 12w_1 - 6w_2$ subject to the constraints

$$4w_1 - 3w_2 \leq 5 \quad ,$$

$$2w_1 - 2w_2 \leq 2 \quad ,$$

$$w_1 - 3w_2 \leq 6 \quad ,$$

$$w_1 \geq 0 \quad , \quad w_2 \geq 0 \quad .$$

This is an even better-looking answer.

Another Way to get the nice-looking answer:

Do much less pre-processing by only changing the second constraint:

Minimize $5\,x_1 \,+\, 2\,x_2 \,+\, 6\,x_3$

subject to

$$4x_1 + 2x_2 + x_3 \geq 12 \quad , \quad -3x_1 - 2x_2 - 3x_3 \geq -6 \quad ,$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0 \quad , \quad x_3 \geq 0 \quad .$$

Now it is **exactly** in the format of the dual. Since the dual of the dual is (equivalent) to the same, the nicer-looking answer is gotten immediately.