Solutions to Attendance Quiz for Lecture 18

1. Solve the following assignment problem

$$\begin{bmatrix} 3 & 2 & 5 & 8 & 9 \\ 6 & 7 & 4 & 2 & 3 \\ 5 & 3 & 5 & 4 & 2 \\ 4 & 7 & 3 & 2 & 4 \\ 2 & 6 & 5 & 5 & 3 \end{bmatrix} \quad .$$

Sol. of 1:

Step 1: We have to make sure that there is at least one zero in each row, getting, an equivalent, but much easier problem, by subtracting from each entry the smallest entry of its row.

- The smallest entry of row 1, is 2, hence it becomes $[3-2, 2-2, 5-2, 8-2, 9-2] = [1, 0, 3, 6, 7]$
- The smallest entry of row 2, is 2, hence it becomes $[6-2, 7-2, 4-2, 3-2, 3-2] = [4, 5, 2, 0, 1]$
- The smallest entry of row 3, is 2, hence it becomes $[5-2, 3-2, 5-2, 4-2, 2-2] = [3, 1, 3, 2, 0]$
- The smallest entry of row 4, is 2, hence it becomes $[4-2, 7-2, 3-2, 2-2, 4-2] = [2, 5, 1, 0, 2]$
- The smallest entry of row 5, is 2, hence it becomes $[2-2, 6-2, 5-2, 5-2, 3-2] = [0, 4, 3, 3, 1]$

(Comment: in this problem, the smallest entry in each row happened to be the same, this is a coincidence, in general they are different.)

Our simplified problem is now

$$\begin{bmatrix} 1 & 0 & 3 & 6 & 7 \\ 4 & 5 & 2 & 0 & 1 \\ 3 & 1 & 3 & 2 & 0 \\ 2 & 5 & 1 & 0 & 2 \\ 0 & 4 & 3 & 3 & 1 \end{bmatrix} \quad .$$

Next, we have to make sure that every **column** has at least one zero. Only the third column needs fixing. The smallest entry is 1, so we subtract 1 from every entry of the 3-rd column, getting an even simpler equivalent problem.

$$\begin{bmatrix} 1 & 0 & 2 & 6 & 7 \\ 4 & 5 & 1 & 0 & 1 \\ 3 & 1 & 2 & 2 & 0 \\ 2 & 5 & 0 & 0 & 2 \\ 0 & 4 & 2 & 3 & 1 \end{bmatrix} \quad .$$

This ends Step 1.

Step 2:

The leftmost (and as it happened, the only) 0 in row 1 is at column 2, so we star that 0, and remember that column 2 is out of the running from now on.

Now it looks like this:

$$\begin{bmatrix} 1 & 0^* & 2 & 6 & 7 \\ 4 & 5 & 1 & 0 & 1 \\ 3 & 1 & 2 & 2 & 0 \\ 2 & 5 & 0 & 0 & 2 \\ 0 & 4 & 2 & 3 & 1 \end{bmatrix} .$$

The leftmost (and as it happened, the only) (still available) 0 in row 2 is at column 4, so we star that 0, and remember that also column 4 is out of the running from now on.

Now it looks like this:

$$\begin{bmatrix} 1 & 0^* & 2 & 6 & 7 \\ 4 & 5 & 1 & 0^* & 1 \\ 3 & 1 & 2 & 2 & 0 \\ 2 & 5 & 0 & 0 & 2 \\ 0 & 4 & 2 & 3 & 1 \end{bmatrix} .$$

The leftmost (and as it happened, the only) (still available) 0 in row 3 is at column 5, so we star that 0, and remember that also column 5 is out of the running from now on.

Now it looks like this:

$$\begin{bmatrix} 1 & 0^* & 2 & 6 & 7 \\ 4 & 5 & 1 & 0^* & 1 \\ 3 & 1 & 2 & 2 & 0^* \\ 2 & 5 & 0 & 0 & 2 \\ 0 & 4 & 2 & 3 & 1 \end{bmatrix} .$$

The leftmost (still available) 0 in row 4 is at column 3, So we star it, and remember that also column 3 is out of the running from now on. Now it looks like this:

$$\begin{bmatrix} 1 & 0^* & 2 & 6 & 7 \\ 4 & 5 & 1 & 0^* & 1 \\ 3 & 1 & 2 & 2 & 0^* \\ 2 & 5 & 0^* & 0 & 2 \\ 0 & 4 & 2 & 3 & 1 \end{bmatrix} .$$

The leftmost (still available) 0 in row 5 is at column 1, so we star it, and we are done! Now it looks like this:

$$\begin{bmatrix} 1 & 0^* & 2 & 6 & 7 \\ 4 & 5 & 1 & 0^* & 1 \\ 3 & 1 & 2 & 2 & 0^* \\ 2 & 5 & 0^* & 0 & 2 \\ 0^* & 4 & 2 & 3 & 1 \end{bmatrix} .$$

We lucked out! Every row and every column has a starred zero, and we are done. No need for Steps 3 and 4. To get the permutation matrix, we replace the 0^* by 1 and all the other entries by 0.

Ans. to 1: The solution is the permutation matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad ,$$

the cost is $c_{12} + c_{24} + c_{35} + c_{43} + c_{51} = 2 + 2 + 2 + 3 + 2 = 11$.

Comment: A more succinct way of expressing the solution is that it is the permutation (in **two-line notation**)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 3 & 1 \end{bmatrix}$$

and an even shorter one is to express it in **one-line notation** 24531.