Solutions to Attendance Quiz for Lecture 17

1. Consider the following transportation problem where $\mathbf{s}$ is the **supply vector**, $\mathbf{d}$ is the **demand vector**, and $\mathbf{C}$ is the **cost matrix** between the supply sites and the demand sites.

$$\mathbf{C} = \begin{bmatrix} 1 & 3 \\ 4 & 8 \end{bmatrix} \quad , \quad \mathbf{s} = \begin{bmatrix} 10 \\ 20 \end{bmatrix} \quad , \quad \mathbf{d} = \begin{bmatrix} 15 \\ 10 \end{bmatrix} \quad .$$

(a) Since the total demand is less than the total supply, create an equivalent problem where the supply equals the demand.

(b) By using the Minimal Cost Rule (**not** Vogel's method!), find an **initial basic feasible tableau** for the problem in (a).

(c) By starting with the basic feasible solution in (b), find the optimal solution. Also find the minimal cost. Make sure that it is the optimal solution by using the optimality criterion.

(d) Use Vogel's method (**not** the Minimal Cost rule!) to find **initial basic feasible tableau** for the problem in (a).

(e) Compare the answers to (c) and (d).

Sol. of 1(a): Since the total supply is 30 and the total demand is 25, we create a "dummy" "store" and assign it 5 units, and make the cost of transportation 0 (it is just staying in the factory). The new problem is

$$\mathbf{C} = \begin{bmatrix} 1 & 3 & 0 \\ 4 & 8 & 0 \end{bmatrix} \quad , \quad \mathbf{s} = \begin{bmatrix} 10 \\ 20 \end{bmatrix} \quad , \quad \mathbf{d} = \begin{bmatrix} 15 \\ 10 \\ 5 \end{bmatrix} \quad .$$

Sol. of 1(b): We start with

$$\begin{bmatrix} 0 & 0 & 0 & | & \mathbf{10} \\ 0 & 0 & 0 & | & \mathbf{20} \\ & & & | & \\ \mathbf{15} & \mathbf{10} & \mathbf{5} & & \end{bmatrix}$$

The cheapest cell in row 1 is cell $[1,3]$. It can accommodate at most 5 units, so we put $x_{13} = 5$ and the remaining $10-5=5$ from row 1 can go to the second-cheapest cell $[1,1]$, that can accommodate it. The situation now is

$$\begin{bmatrix} 5 & 0 & 5 & | & \mathbf{0} \\ 0 & 0 & 0 & | & \mathbf{20} \\ & & & | & \\ \mathbf{10} & \mathbf{10} & \mathbf{0} & & \end{bmatrix}$$

The cheapest cell in row 2 is [2, 3] but it can't take anything (since the available demand in column 3 is zero), so we put 10 in the second-cheapest cell, [2, 1] and the rest in cell [2, 2].

So we have

$$\begin{bmatrix} 5 & 0 & 5 & | & \mathbf{0} \\ 10 & 10 & 0 & | & \mathbf{0} \\ & & & | & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & \end{bmatrix} \quad .$$

Ans. to 1(b): The initial basic feasible solution using the Minimal Cost Rule is

$$\begin{bmatrix} 5 & 0 & 5 \\ 10 & 10 & 0 \end{bmatrix}$$

(Note that its cost is $5 \cdot 1 + 5 \cdot 0 + 10 \cdot 4 + 10 \cdot 8 = 125$)

Sol. to 1(c): The set of **basic variables** is

$$Basic = \{x_{11}, x_{13}, x_{21}, x_{22}\} \quad ,$$

and the set of **non-basic variables** is

$$NonBasic = \{x_{12}\, ,\, x_{23}\} \quad ,$$

The **dual variables** are: v_1, v_2, w_1, w_2, w_3.

Recall that the set of equations is obtained by setting up :

$v_i + w_j = c_{ij}$ for each basic variable x_{ij} .

We get the following set

$$x_{11} : v_1 + w_1 = 1 \quad ,$$

$$x_{13} : v_1 + w_3 = 0 \quad ,$$

$$x_{21} : v_2 + w_1 = 4 \quad ,$$

$$x_{22} : v_2 + w_2 = 8 \quad .$$

Setting $w_1 = 0$, and solving the system by plugging-in, we easily get

$$v_1 = 1 \quad , \quad v_2 = 4 \quad ,$$

$$w_1 = 0 \quad , \quad w_2 = 4 \quad , \quad w_3 = -1 \quad .$$

Plugging these values into the non-basic cells

$$x_{12} : v_1 + w_2 - c_{12} = 1 + 4 - 3 = 2 \quad ,$$

$$x_{23} : v_2 + w_3 - c_{23} = 4 + (-1) - 0 = 3 \quad .$$

Since x_{23} gives the **largest** value, it is the **entering variable**.

The only horizontal-vertical alternating cycle starting at $[2,3]$, and going through basic cells is

$$[2,3] \to [2,1] \to [1,1] \to [1,3] \to [2,3] \quad .$$

We have $x_{21} = 10$ and $x_{13} = 5$. Since 5 is smaller, the **departing variable** is x_{13} and we update the values of the cells in that cycle by adding 5 to the odd-indexed cells and subtracting 5 from the even-indexed cells

$$x_{23} \leftarrow x_{23} + 5 = 0 + 5 = 5 \quad , \quad x_{21} \leftarrow x_{21} - 5 = 10 - 5 = 5 \quad ,$$

$$x_{11} \leftarrow x_{11} + 5 = 5 + 5 = 10 \quad , \quad x_{13} \leftarrow x_{13} - 5 = 5 - 5 = 0 \quad .$$

The other two entries stay the same. We now have the cheaper solution

$$\begin{bmatrix} 10 & 0 & 0 \\ 5 & 10 & 5 \end{bmatrix} \quad ,$$

whose cost is $10 \cdot 1 + 5 \cdot 4 + 10 \cdot 8 + 5 \cdot 0 = 110$, cheaper by 15 then the initial solution.

We finished **one** iteration of the algorithm. Let's move on to the second iteration.

Now the set of **basic variables** is

$\{x_{11} , x_{21} , x_{22} , x_{23}\}$ and the set of **non-basic variables** is $\{x_{12} , x_{13}\}$.

The dual variables are v_1, v_2, w_1, w_2, w_3, and the equations are

$x_{11} : v_1 + w_1 = 1 \quad ,$

$x_{21} : v_2 + w_1 = 4 \quad ,$

$x_{22} : v_2 + w_2 = 8 \quad .$

$x_{23} : v_2 + w_3 = 0 \quad .$

Setting $v_1 = 0$ and solving, we easily get

$$v_1 = 0 \quad , \quad v_2 = 3 \quad ,$$

$$w_1 = 1 \quad , \quad w_2 = 5 \quad , \quad w_3 = -3 \quad .$$

Plugging these values into the non-basic cells

$$x_{12} : v_1 + w_2 - c_{12} = 0 + 5 - 3 = 2 \quad ,$$

$$x_{13} : v_1 + w_3 - c_{13} = 0 - 3 - 0 = -3 \quad .$$

Since x_{12} gives the **largest** value, it is the **entering variable**.

The only horizontal-vertical alternating cycle starting at $[1, 2]$ and going through basic cells is

$$[1, 2] \to [2, 2] \to [2, 1] \to [1, 1] \to [1, 2] \quad .$$

We have $x_{22} = 10$ and $x_{11} = 10$.

Since x_{22} and x_{11} have the same value, either of them can be taken as the departing variable. (this case is called *degenerate* but it does not matter for the algorithm.).

We update the values of the cells in that cycle by adding 10 to the odd-indexed cells and subtracting 10 from the even-indexed cells

$$x_{12} \leftarrow x_{12} + 10 = 0 + 10 = 10 \quad , \quad x_{22} \leftarrow x_{22} - 10 = 10 - 10 = 0 \quad ,$$

$$x_{21} \leftarrow x_{21} + 10 = 5 + 10 = 15 \quad , \quad x_{11} \leftarrow x_{11} - 10 = 10 - 10 = 0 \quad .$$

The other two entries stay the same. We now have

$$\begin{bmatrix} 0 & 10 & 0 \\ 15 & 0 & 5 \end{bmatrix}$$

(Note that the cost now is $10 \cdot 3 + 15 \cdot 4 + 5 \cdot 0 = 90$).

This ends the **second iteration**. We now move on to the third iteration.

Can we do better? Since there are only three non-zero cells, and we need four basic variables, we pick one of the 0-cells to be basic. Let's pick x_{11}. We have that the set of basic variables is $\{x_{11}, x_{12}, x_{21}, x_{23}\}$ and the set of **non-basic variables** is $\{x_{13}.x_{22}\}$.

The set of equations is:

$x_{11} : v_1 + w_1 = 1 \quad ,$

$x_{12} : v_1 + w_2 = 3 \quad ,$

$x_{21} : v_2 + w_1 = 4 \quad ,$

$x_{23} : v_2 + w_3 = 0 \quad .$

Setting $v_2 = 0$, and solving we easily get

$$v_1 = -3 \quad , \quad v_2 = 0 \quad ,$$

$$w_1 = 4 \quad , \quad w_2 = 6 \quad , \quad w_3 = 0 \quad .$$

Plugging these values into the non-basic cells

$$x_{13} : v_1 + w_3 - c_{13} = -3 + 0 - 0 = -3 \quad , \quad x_{22} : v_2 + w_2 - c_{22} = 0 + 6 - 8 = -2 \quad .$$

Since these are all **non-positive** (in fact, in this case, all negative) we have **arrived at our destination**. And this is indeed an optimal solution.

Ans. to 1(c) An optimal solution is

$$\begin{bmatrix} 0 & 10 & 0 \\ 15 & 0 & 5 \end{bmatrix} \quad ,$$

and the minimal cost is 90.

Sol. to 1(d) At the very beginning there is nothing assigned, and we have

$$\begin{bmatrix} 0 & 0 & 0 & | & \mathbf{10} \\ 0 & 0 & 0 & | & \mathbf{20} \\ & & & | & \\ \mathbf{15} & \mathbf{10} & \mathbf{5} & & \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} 1 & 3 & 0 & | & \mathbf{1} \\ 4 & 8 & 0 & | & \mathbf{4} \\ & & & & \\ \mathbf{3} & \mathbf{5} & \mathbf{0} & & \end{bmatrix} \quad ,$$

The largest difference is at column 2, and the cheapest cell is $[1, 2]$.

Since the demand for column 2 is the same as the supply of row 1, we can cross either of them (**but not both!**). It is better to cross the first row, since then we are only left with one row.

Now the situation now is

$$\begin{bmatrix} 0 & 10 & 0 & | & \mathbf{0} \\ 0 & 0 & 0 & | & \mathbf{20} \\ & & & | & \\ \mathbf{15} & \mathbf{0} & \mathbf{5} & & \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} \underline{1} & \underline{3} & \underline{0} & | & \mathbf{na} \\ 4 & 8 & 0 & | & \mathbf{4} \\ & & & & \\ \mathbf{na} & \mathbf{na} & \mathbf{na} & & \end{bmatrix} \quad ,$$

There is only one row left, and we assign 5 to cell $[2, 3]$ (the cheapest) and the remaining 15 to cell $[2, 1]$. Getting that the **initial basic feasible solution** using **Vogel's method** is

$$\begin{bmatrix} 0 & 10 & 0 \\ 15 & 0 & 5 \end{bmatrix} \quad .$$

This looks familiar!

Sol. to 1(e): Using Vogel's method we may get the optimal solution right away!

Moral: In real life, save yourself some unnecessary work by using Vogel's method to get a good **head-start** by picking an initial basic feasible solution using Vogel's method rather than the simpler Minimal Cost Rule.

Of course, unless you are specifically asked to use the minimal cost rule, then you have to obey the instructions, unless you want to get **no credit**.