Solutions to Attendance Quiz for Lecture 11

1. By using the initial tableau for the following problem

Minimize $x_1 + x_2$ subject to the constraints

$$2x_1 + x_2 \geq 2 \quad , \quad x_1 + 2x_2 \geq 2 \quad ,$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0 \quad ,$$

that we got in attendance quiz 10, find the optimal solution using the big-M method.

$BASIC$	x_1	x_2	x_3	x_4	y_1	y_2	z	RHS
y_1	2	1	-1	0	1	0	0	2
y_2	1	2	0	-1	0	1	0	2
	$1-3M$	$1-3M$	M	M	0	0	1	$-4M$

.

Sol. to 1

Both the first and second column may be taken as the pivot column, since they are equal and most negative. (Recall that M is a **fixed** but HUGE positive number).

Let's take the first column.

Let's compute the θ-ratios of the rows.

- The θ-ratio of the first row is $2/2 = 1$;

- The θ-ratio of the second row is $2/1 = 2$.

Since the first row is the smallest, the **pivot row** is the first row, and the **pivot entry** is the $(1,1)$ entry that happens to be 2. It follows that the **entering** variable is x_1 and the **departing** basic variable is y_1. Let's first make the pivot entry 1 as it should by doing $\frac{1}{2}r_1 \rightarrow r_1$. Let's also indicate the entering and departing basic variables by by arrows.

$BASIC$	$x_1^{\downarrow}$	x_2	x_3	x_4	y_1	y_2	z	RHS
${}^{\leftarrow}y_1$	1	$1/2$	$-1/2$	0	$1/2$	0	0	1
y_2	1	2	0	-1	0	1	0	2
	$1-3M$	$1-3M$	M	M	0	0	1	$-4M$

.

In order to make x_1 a basic variable we need to make all the entries in its column (except for the pivot of course) 0. To accomplish this, we perform the elementary row operations

$r_2 - r_1 \to r_2$ and $r_3 + (3M-1)r_1 \to r_3$. We get the new tableau, and we also replace the departing y_1 in the basic variables column by the entering x_1

$BASIC$	x_1	x_2	x_3	x_4	y_1	y_2	z	RHS
x_1	1	$1/2$	$-1/2$	0	$1/2$	0	0	1
y_2	0	$3/2$	$1/2$	-1	$-1/2$	1	0	1
	0	$1/2 - 3M/2$	$1/2 - M/2$	M	$3M/2 - 1/2$	0	1	$-M-1$

.

Now the most negative entry in the last row is the one belonging to the x_2 column, hence the **entering** basic variable is x_2. Who is the **departing** basic variable?

Let's compute the θ-ratios of the rows.

- The θ-ratio of the first row is $1/(1/2) = 2$;
- The θ-ratio of the second row is $1/(3/2) = 2/3$.

Since the second row has the smallest θ ratio, it is now the **pivot** row, and the $(2,2)$ entry is the pivot entry. Also y_2 is the **departing** basic variable.

We first need to make the pivot entry 1 by doing $\frac{2}{3}r_2 \to r_2$. At the same time, let's indicate the entering and departing variables by arrows.

$BASIC$	x_1	$x_2^{\downarrow}$	x_3	x_4	y_1	y_2	z	RHS
x_1	1	$1/2$	$-1/2$	0	$1/2$	0	0	1
${}^{\leftarrow}y_2$	0	1	$1/3$	$-2/3$	$-1/3$	$2/3$	0	$2/3$
	0	$-\frac{3}{2}M + \frac{1}{2}$	$-\frac{1}{2}M + \frac{1}{2}$	M	$\frac{3}{2}M - \frac{1}{2}$	0	1	$-M-1$

.

In order to make x_2 a basic variable, we need to make every entry, except the pivot, equal to 0. This is accomplished via the row operations

$r_1 - \frac{1}{2}r_2 \to r_1$ and $r_3 + \frac{3M-1}{2}r_2 \to r_3$.

We get the new tableau (we also replace the departing y_2 in the basic variables column by the entering x_2

$BASIC$	x_1	x_2	x_3	x_4	y_1	y_2	z	RHS
x_1	1	0	$-2/3$	$1/3$	$2/3$	$-1/3$	0	$2/3$
x_2	0	1	$1/3$	$-2/3$	$-1/3$	$2/3$	0	$2/3$
	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$M - \frac{1}{3}$	$M - \frac{1}{3}$	1	$-\frac{4}{3}$

.

At long last, there are no more negative entries in the bottom (goal) row and we are done!

The optimal solution, is $x_1 = \frac{2}{3}$, $x_2 = \frac{2}{3}$ and the optimal value is $-\frac{4}{3}$. But this is the optimal value of the converted problem to maximize. Going back to the original, minimization problem, the optimal value is $\frac{4}{3}$.

Ans. to 1: The optimal solution is $x_1 = \frac{2}{3}$ $x_2 = \frac{2}{3}$ and the optimal value is $\frac{4}{3}$.

Comment: It is much easier to do this particular problem with either the graphical method, and even with the more tedious algebraic method of section 1.5. But for larger problems we have to use the simplex method, and of course, computers do all the heavy lifting. The point of doing it this way is that you will **understand** this algorithm, and working out a small example by hand will (hopefully) do the job.