Consider the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{4}$$

Multiplying out the left-hand side, we obtain

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{5}$$

Using the properties of matrix addition and scalar multiplication, we may rewrite (5) as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Thus, if (4) is written in compact form as

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

and $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ are the columns of $\mathbf{A}$, then we have shown that (4) can be written as

$$x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_n\mathbf{A}_n = \mathbf{b}. \tag{6}$$

That is, writing the matrix equation $\mathbf{Ax} = \mathbf{b}$ is equivalent to writing $\mathbf{b}$ as a linear combination of the columns of $\mathbf{A}$. Furthermore, if $\mathbf{x}$ is a solution to $\mathbf{Ax} = \mathbf{b}$, then the components of $\mathbf{x}$ are the coefficients of the columns of $\mathbf{A}$ when $\mathbf{b}$ is written as a linear combination of these columns.

An important application of this discussion occurs in the case when $\mathbf{A}$ is nonsingular. In this case, the columns of $\mathbf{A}$ form a linearly independent set of n vectors in R^n. Thus, this set is a basis for R^n, and $\mathbf{b}$ can be uniquely written as a linear combination of the columns of $\mathbf{A}$. The solution to $\mathbf{Ax} = \mathbf{b}$ gives the coordinate vector of $\mathbf{b}$ with respect to this basis. That is, from Equation (6), the coordinates of $\mathbf{b}$ with respect to the basis $\{\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n\}$ are $x_1, x_2, \ldots, x_n$. Since $\mathbf{A}$ is nonsingular, $\mathbf{A}^{-1}$ exists, and we may write $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. The columns of $\mathbf{A}^{-1}$ also form a basis for R^n. In the same manner as that for $\mathbf{b}$, we can obtain the coordinates of $\mathbf{x}$ with respect to the basis of columns of $\mathbf{A}^{-1}$. These coordinates are the components of $\mathbf{b}$.

Definition. The **rank** of an $m \times n$ matrix $\mathbf{A}$ is the number of nonzero rows in the matrix in reduced row echelon form that is row equivalent to $\mathbf{A}$.

Example 9. Find the rank of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 1 & 2 \\ 2 & 3 & -2 & 5 & 2 \\ 4 & 7 & -4 & 7 & 6 \\ 3 & 4 & -3 & 9 & 2 \\ -1 & 0 & 1 & -7 & 2 \end{bmatrix}.$$

Solution. Transforming $\mathbf{A}$ to reduced row echelon form we find (verify) that $\mathbf{A}$ is row equivalent to

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & -1 & 7 & -2 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, the rank of $\mathbf{A}$ is 2. △

0.5 EXERCISES

1. Let

$$S = \left\{ \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} \right\}.$$

Which of the following vectors are linear combinations of the vectors in S?

(a) $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} -6 \\ 2 \\ 4 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2. Let

$$S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -8 \\ -10 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Which of the following vectors are linear combinations of the vectors in S?

(a) $\begin{bmatrix} -2 \\ 0 \\ 6 \\ 6 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 3 \\ 1 \\ 0 \\ -2 \end{bmatrix}$

3. Which of the following sets of vectors span R^2?

(a) $\left\{\begin{bmatrix}2\\-1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}2\\3\end{bmatrix}\right\}$ (d) $\left\{\begin{bmatrix}-1\\3\end{bmatrix}, \begin{bmatrix}2\\-6\end{bmatrix}\right\}$

4. Which of the following sets of vectors span R^3?

(a) $\left\{\begin{bmatrix}3\\2\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\-1\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}3\\-1\\2\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

5. Which of the following sets of vectors in R^3 are linearly dependent? For those that are, express one vector as a linear combination of the rest.

(a) $\left\{\begin{bmatrix}3\\2\\-1\end{bmatrix}, \begin{bmatrix}-1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}3\\4\\-2\end{bmatrix}, \begin{bmatrix}5\\7\\-1\end{bmatrix}\right\}$ (d) $\left\{\begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}4\\5\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix}\right\}$

6. Follow the directions of Exercise 5 for the following sets of vectors in R^4.

(a) $\left\{\begin{bmatrix}-1\\2\\4\\3\end{bmatrix}, \begin{bmatrix}3\\-1\\2\\5\end{bmatrix}, \begin{bmatrix}-5\\5\\6\\1\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}2\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\2\end{bmatrix}, \begin{bmatrix}8\\6\\4\\6\end{bmatrix}, \begin{bmatrix}3\\2\\0\\1\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}3\\1\\2\\2\end{bmatrix}, \begin{bmatrix}2\\1\\3\\1\end{bmatrix}, \begin{bmatrix}2\\1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}$ (d) $\left\{\begin{bmatrix}0\\0\\1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\2\end{bmatrix}\right\}$

7. Which of the sets in Exercise 3 are bases for R^2?
8. Which of the sets in Exercise 4 are bases for R^3?
9. Which of the sets in Exercise 5 are bases for R^3?
10. Which of the sets in Exercise 6 are bases for R^4?
11. Which of the following sets form a basis for R^3? Express the vector

$$\begin{bmatrix}-3\\0\\1\end{bmatrix}$$

as a linear combination of the vectors in each set that is a basis.

(a) $\left\{\begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}4\\-2\\1\end{bmatrix}, \begin{bmatrix}0\\-8\\-7\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}-3\\-1\\2\end{bmatrix}, \begin{bmatrix}3\\4\\1\end{bmatrix}, \begin{bmatrix}-3\\-2\\1\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

12. Let

$$S = \left\{\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$$

be a basis for R^2. Find the coordinate vector $[\mathbf{x}]_S$ of the indicated vector $\mathbf{x}$ with respect to S.

(a) $\mathbf{x} = \begin{bmatrix}3\\4\end{bmatrix}$ (b) $\mathbf{x} = \begin{bmatrix}-2\\1\end{bmatrix}$ (c) $\mathbf{x} = \begin{bmatrix}1\\1\end{bmatrix}$ (d) $\mathbf{x} = \begin{bmatrix}1\\2\end{bmatrix}$

13. Let

$$S = \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$$

be a basis for R^3. Find the coordinate vector $[\mathbf{x}]_S$ of the indicated vector $\mathbf{x}$ with respect to S.

(a) $\mathbf{x} = \begin{bmatrix}1\\0\\1\end{bmatrix}$ (b) $\mathbf{x} = \begin{bmatrix}-1\\2\\3\end{bmatrix}$ (c) $\mathbf{x} = \begin{bmatrix}2\\1\\1\end{bmatrix}$ (d) $\mathbf{x} = \begin{bmatrix}3\\2\\1\end{bmatrix}$

14. Suppose that S_1 and S_2 are finite sets of vectors in R^n and that S_1 is a subset of S_2. Prove the following.
(a) If S_1 is linearly dependent, so is S_2.
(b) If S_2 is linearly independent, so is S_1.
15. Show that any set of vectors in R^n that includes the zero vector must be linearly dependent.
16. Show that any set of $n + 1$ vectors in R^n must be linearly dependent.
17. Show that R^n cannot be spanned by a set containing fewer than n vectors.
18. Find the rank of the matrix

$$\begin{bmatrix}2 & 3 & 4 & -1 & 2\\ 4 & -1 & 6 & -7 & -6\\ 3 & 2 & -1 & 3 & 4\end{bmatrix}. \quad \boxed{3}$$

19. Find the rank of the matrix

$$\begin{bmatrix}1 & 3 & -2 & 2\\ 2 & 3 & 4 & 1\\ 3 & 6 & 2 & 3\\ 1 & 6 & -10 & 5\end{bmatrix}.$$