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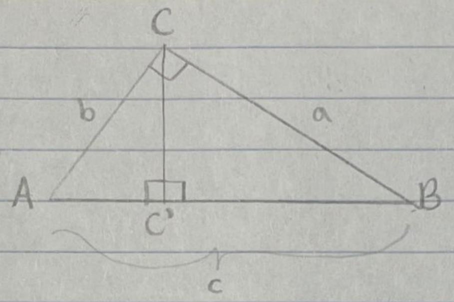
Exam 2

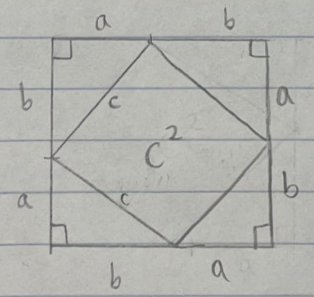
① Proof #1:



Thus $\triangle ACC'$, $\triangle CBC'$, and $\triangle ABC$ are similar because they have the same angles. Then we know that there exists a nonzero constant, call it α, where the area of $\triangle ACC' = \alpha b^2$, the area of $\triangle CBC' = \alpha a^2$, and the area of $\triangle ABC = \alpha c^2$. Then we use the fact that (area of $\triangle ACC'$) + (area of $\triangle CBC'$) = (area of $\triangle ABC$), so we have that

$$\alpha a^2 + \alpha b^2 = \alpha c^2, \text{ thus } a^2+b^2=c^2.$$

Proof #2:



This is an $(a+b) \times (a+b)$ square.

In the middle, there is a $c \times c$ square with area of c^2. The area of the entire (larger) square can be found by adding up the area of the four triangles, which is $4 \cdot (\frac{1}{2}ab)$, and adding that to c^2 to get $c^2 + 4\cdot(\frac{1}{2}ab) = c^2 + 2ab$. This is equal to the area of the big square, which is $(a+b)\times(a+b) = a^2+2ab+b^2$. Equating these, we see that

$$a^2+2ab+b^2 = c^2+2ab, \text{ so } a^2+b^2=c^2.$$

(2) For the sake of contradiction, suppose that $3^{1/7}$ is rational. Then $3^{1/7}$ can be written as

$$3^{1/7} = \frac{m}{n} \quad \text{for some positive integers } m, n$$

We raise both sides of this equation to the seventh power and transpose to obtain

$$m^7 = 3n^7$$

Next we use a lemma that states: For any integer n, the exponent of all the primes in the prime decomposition of n^7 are multiples of 7.

Thus if $n = p_1^{a_1} \cdots p_k^{a_k}$ where $p_1, \ldots, p_k$ are prime, then

$$n^7 = p_1^{7a_1} \cdots p_k^{7a_k}$$

We see that in the equation $m^7 = 3n^7$, that the exponent of 3 on the right hand side is of the form $7b+1$ (which is 1 mod 7), whereas the exponent of 3 on the left hand side is a multiple of 7, which is of the form $7a$.

We arrive at a contradiction, so $3^{1/7}$ must be irrational.

③ a.) Pascal's Triangle

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}
$$

Pascal's Triangle Mod 2

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 0 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \\
1 \quad 0 \quad 0 \quad 0 \quad 1 \\
1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \\
1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1
\end{array}
$$

Even number in original triangle becomes 0 in mod 2 version
Odd number in original triangle becomes 1 in mod 2 version.

I highlighted the middle 0 section in green. This shows that the remaining parts are three identical triangles in pink that span 4 rows each. This is because the fractal diagram is

$$
\begin{array}{ccc}
 & A(n-1) & \\
A(n-1) & 0 & A(n-1)
\end{array}
\quad \text{with } A(n) = \text{this array}
$$

b.) Feigenbaum's constant is related to the logistic map $x_{n+1} = k x_n (1-x_n)$. When $k<1$, the population will go extinct, in other words, it goes to 0. When $1<k<3$, eventually the population will stabilize (tend to a single number). When $k>3$, for a while, there will be a period of 2 (fluctuating between 2 values). If we make k even bigger, we will get a period of 4 (fluctuate between 4 values). Again, making k even bigger, we will get a period of 8 (fluctuate between 8 values). Feigenbaum used r and saw that r_k tells you the transition from period 2^{k-1} to 2^k. Thus

$$\lim_{n \to \infty} \frac{r_{k+1} - r_k}{r_k - r_{k-1}}$$ exists and the ratios go to Feigenbaum's constant, which is approx. 4.669

4 a.) A Platonic solid is a polyhedron that has all its faces as identical perfect polygons.

b.) We know that every vertex has a edges that comes out from it, and there are v vertices, so there must be aV edges. But we also know that every edge belongs to 2 vertices, so we overcounted. Thus $2E = aV$, so $E = \frac{aV}{2}$ and thus $\boxed{V = \frac{2E}{a}}$

Similarly, we know that every face has b edges around it, and there are F faces, so there must be bF edges. But we also know that every edge belongs to 2 faces, so we overcounted. Thus $2E = bF$, so $E = \frac{bF}{2}$, and thus $\boxed{F = \frac{2E}{b}}$

c.) We plug in $V = \frac{2E}{a}$ and $F = \frac{2E}{b}$ into $V - E + F = 2$ to get $\frac{2E}{a} - E + \frac{2E}{b} = 2$. Solving for E, we get $E\left(\frac{2}{a} - 1 + \frac{2}{b}\right) = 2$, so $E = \frac{2}{\frac{2}{a} - 1 + \frac{2}{b}}$. Using this new formula for E, we plug into $F = \frac{2E}{b}$ to obtain

$$F = \frac{\left(2 \cdot \frac{2}{\frac{2}{a} - 1 + \frac{2}{b}}\right)}{b} = \frac{\frac{4}{\frac{2}{a} - 1 + \frac{2}{b}}}{b} = \frac{4}{\frac{2b}{a} - b + 2}$$

d.) $a = 3, b = 3 \to$ Tetrahedron, $F = \frac{4}{\frac{2\cdot 3}{3} - 3 + 2} = 4$

$a = 3, b = 4 \to$ Hexahedron, $F = \frac{4}{\frac{2\cdot 4}{3} - 4 + 2} = \frac{4}{\frac{8}{3} - \frac{6}{3}} = \frac{4}{\frac{2}{3}} = 6$

$a = 3, b = 5 \to$ Dodecahedron, $F = \frac{4}{\frac{2\cdot 5}{3} - 5 + 2} = \frac{4}{\frac{10}{3} - \frac{9}{3}} = \frac{4}{\frac{1}{3}} = 12$

$a = 4, b = 3 \to$ Octahedron, $F = \frac{4}{\frac{2\cdot 3}{4} - 3 + 2} = \frac{4}{\frac{6}{4} - \frac{4}{4}} = \frac{4}{\frac{1}{2}} = 8$

$a = 4, b = 4 \to$ does not make sense

$a = 4, b = 5 \to$ does not make sense

$a = 5, b = 3 \to$ Icosahedron, $F = \frac{4}{\frac{2\cdot 3}{5} - 3 + 2} = \frac{4}{\frac{6}{5} - \frac{5}{5}} = \frac{4}{\frac{1}{5}} = 20$

$a = 5, b = 4 \to$ does not make sense

$a = 5, b = 5 \to$ does not make sense

This proves that there are exactly 5 Platonic solids.

⑤ Let the set G have m elements and let the set H have n elements. Then we can say that $H = \{h_1, \ldots, h_n\}$, where we can assume that all of the elements $h_1, \ldots, h_n$ are distinct. Next, if we let $G = H$, then we see that $\frac{m}{n} = 1$, so that takes care of that case.

Now suppose that $G \neq H$. Then there must exist an element, call it g_1, where $g_1 \in G \setminus H$. We consider the left coset $g_1 H = \{g_1 h_1, \ldots, g_1 h_n\}$. Here, the elements $g_1 h_1, \ldots, g_1 h_n$ are all distinct, because if we suppose they are not for the sake of contradiction, then for some $1 \leq i < j \leq n$, we get $g_1 h_i = g_1 h_j$. Multiplying both sides on the left by g_1^{-1}, we obtain $h_i = h_j$, which is the contradiction since we said before that $h_1, \ldots, h_n$ are distinct. Also, it must be that $g_1 H$ and H have no elements in common. This is because if we suppose not for the sake of contradiction, then for some h_i, h_j in H, we get $g_1 h_i = h_j$, and multiplying both sides on the right by h_i^{-1}, we get $g_1 = h_j h_i^{-1}$. This is a contradiction since H is a group so g_1 would be in H, which is false.

Now let $G = H \cup g_1 H$. This satisfies the condition. If not, let $g_2 \in G$ such that $g_2 \notin H$ and $g_2 \notin g_1 H$. Using the same argument as above, all of $g_2 H$'s elements are distinct and have nothing in common with H and $g_1 H$. Continue doing this until we arrive at

$$G = H \cup g_1 H \cup g_2 H \cup \cdots \cup g_{r-1} H$$

Each set has n elements with nothing in common, so $m = nr$, thus $r = \frac{m}{n}$ is an integer.

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⑥ They are called the Cauchy-Riemann equations. They are special because when Riemann mapped the (xy)-plane conformally on the (uv)-plane, he showed that there exists a function that can transform any simply connected region in one plane into any simply connected region in the other plane. This theory of complex functions led to the Riemann surface, which incorporated topology into analysis.

⑦ William Rowan Hamilton discovered the quaternions. He lived in Dublin.

⑧ Heron's formula is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where A is the area of a triangle. He lived in the first century A.D.

⑨ Newton studied at Cambridge. His teacher was Isaac Barrow. Barrow yielded his professorship to Newton. Newton stayed at Cambridge until 1696 when he became warden. Later on, he became master of the mint.

⑩ He was born in Leipzig. He spent most of his life near the Court of Hanover. King George I was once the employer of Leibniz.

⑪ a.) Viete's infinite product was

$$\frac{2}{\pi} = \cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16}\cos\frac{\pi}{32}\cdots$$

b.) John Napier and Henry Briggs initiated the use of logarithms.

(12) a.) A Eulerian path in a graph is a route that starts at some vertex and ends at another vertex, and each edge is visited exactly once.

b.) The necessary condition is that every vertex in the graph, except for 2, has an even degree, and the path must start and end at one of the vertices that has an odd degree.

c.) Let the starting vertex of the path be called a. When the route leaves a, that action adds 1 to the degree of a. Then, when the route goes through another vertex along the way, that action adds 2 to that vertex's degree, since it enters and leaves the vertex. Then, when the route finishes at the ending vertex, call it b, that action adds 1 to the degree of b. This demonstrates the necessary condition, because all the vertices except two will have even degree, whereas a and b will have an odd degree. Also, we demonstrated that the route began at a and ended at b, which are the two vertices with odd degree.