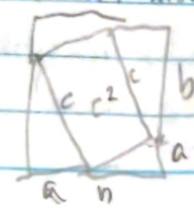
Sarah Catonic — Exam 2 — 12/6/2021

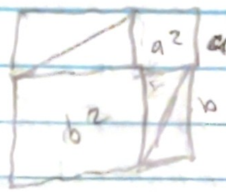
1. The Rearrangement Proof

Start off with a square of area $(a+b)^2$



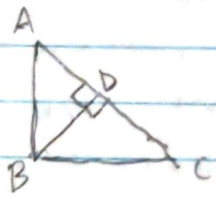
the center portion has area c^2.

Rearrange this such that it is in this formation



Since the area of the 4 triangles hasn't changed, then it must be that $a^2+b^2=c^2$, thus proving the Pythagorean theorem.

Similar Triangles Proof



Draw a right triangle ABC. Draw a perpendicular segment to AC through B. Label the point where the perpendicular segment hits AC, D.

$\triangle ADB$ is similar to $\triangle ABC$ because they both share angle A, and $\angle ADB$ and $\angle ABC$ are right angles. Since two of their angles are congruent, they are similar. $\triangle BDC$ is similar to $\triangle ABC$ because they both share $\angle C$ and $\angle CDB$ and $\angle BDC$ are both right angles. Since two of their angles are congruent, they are similar.

So, by $\triangle ADB$ being similar to $\triangle ABC$ and $\triangle BDC$ similar to $\triangle ABC$

$$\frac{AB}{AC} = \frac{AD}{AB} \qquad \frac{BC}{AC} = \frac{BD}{BC}$$

so $AB^2 = AD \times AC$ and $BC^2 = AC \times BD$

So $A^2 + B^2 = C^2$

2. Assume to the contrary that $\sqrt[7]{3}$ is rational, so there exists $m, n \in \mathbb{Z}$ such that $\sqrt[7]{3} = \frac{m}{n}$, with m and n in lowest terms. Raising both sides to the 7th power

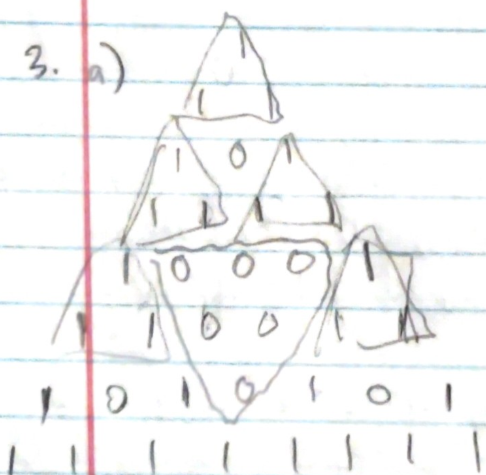
$$3 = \frac{m^7}{n^7} \Rightarrow 3n^7 = m^7$$

So, m^7 is divisible by 3, which means m is divisible by 3. So m can be written in the form $m = 3k$ for some $k \in \mathbb{Z}$.

So, $3n^7 = 3^7 m^7 \Rightarrow n^7 = 3^6 m^7$

So, n^7 is divisible by 3, which means n is divisible by 3. However, we assumed m and n were in lowest terms, but they are both divisible by 7, so this is false. Therefore, our original assumption that $\sqrt[7]{3}$ is rational must be false, so $\sqrt[7]{3}$ is irrational.

3. a)



b) The Feigenbaum constant is the ratio of the differences between k values in the equation $x_{n+1} = kx_n(1-x_n)$ that leads to the doubling of the period. It is approximately 4.669...

4. a) A Platonic solid is a closed polyhedron where each face is the same shape and the number of edges coming out of each vertex is the same for all vertices.

b) $aV = 2E$ so $V = \frac{2E}{a}$ since every edge connects two vertices and every vertex has a edges coming out of it.

$bF = 2E$ so $F = \frac{2E}{b}$ since every edge connects two faces and there are b edges around every face.

c) $E = \frac{bF}{2}$ so $V = \frac{2\left(\frac{bF}{2}\right)}{a} = \frac{bF}{a}$

$V - E + F = 2 \Rightarrow \frac{bF}{a} - \frac{bF}{2} + F = 2$

$F\left(\frac{b}{a} - \frac{b}{2} + 1\right) = 2$

so $F = 2 / \left(\frac{b}{a} - \frac{b}{2} + 1\right)$

d) $3 \leq a, b \leq 5$

$a = 3, b = 3$ $\quad F = \frac{2}{1 - \frac{3}{2} + 1} = \frac{2}{1/2} = 4$ Tetrahedron

$a = 3, b = 4$ $\quad F = \frac{2}{\frac{4}{3} - \frac{3}{2} + 1} = 6$ cube

$a = 4, b = 3$ $\quad \frac{2}{\frac{3}{4} - \frac{3}{2} + 1} = 8$ octahedron

$a = 3, b = 5$ $\quad \frac{2}{\frac{5}{3} - \frac{3}{2} + 1} = 12$ dodecahedron

$a = 5, b = 3$ $\quad \frac{2}{\frac{3}{5} - \frac{5}{2} + 1} = 20$ icosahedron

5. The left cosets [illegible] have the same number of elements. If any cosets share an element, then they must be the same coset. Since all cosets are distinct and the cosets all make up the group, and each of the cosets have the same number of elements. Then, the order of each coset is the same. So, $\frac{|G|}{|H|}$ is an integer.

6. Euler $\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0$

7. William Rowan Hamilton in Dublin, Ireland

8. Formula for the area of a triangle $A = \sqrt{s \cdot (s-a)(s-b)(s-c)}$
3rd century BC

9. Cambridge; Johann Bernoulli; fought in a duel; farmer

10. London, Paris ; Fredrick the Great

11. a) $\frac{2}{\pi} = \cos\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{8}\right) \cos\left(\frac{\pi}{16}\right) \cos\left(\frac{\pi}{32}\right)$

b) John Neper and Henry Briggs

12. a) A Eulerian path is one in which you cross all edges once.

b) In order for a path to be Eulerian, exactly two vertices must have odd degree, or an odd number of edges coming out of it. All other vertices must have even degree, or an even number of edges coming out of them. A Eulerian cycle has each vertex having even degree.

c) When entering a vertex via an edge, you must exit by another. The first and last vertex have odd degree because you leave by 1 and then at the last step you return by 1. All others have even degree because when you enter a vertex you must leave by another.