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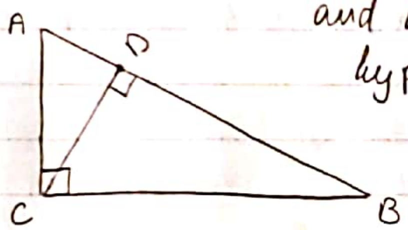
Midterm 2

12/06/2021

1. Two proofs of the Pythagorean theorem.

(1) Geometric proof using similar triangles.
Construct a right triangle as following:
and draw a perpendicular line to the hypotenuse.



Note that the smaller triangles, $\triangle ADC$ and $\triangle CDB$ are similar to $\triangle ACB$ because all of their angles are equal. Use this to construct ratios:

1) Since $\triangle ADC \sim \triangle ACB$

$$\frac{AD}{AC} = \cos \angle A = \frac{AC}{AB}$$

Cross multiply to get $|AD|\,|AB| = |AC|^2$

2) Since $\triangle CDB \sim \triangle ACB$

$$\frac{DB}{CB} = \cos \angle B = \frac{CB}{AB}$$

Cross multiply to get $|DB|\,|AB| = |CB|^2$

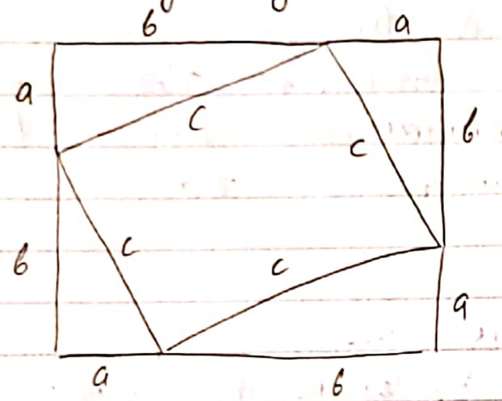
Now add the two equations from 1) and 2):

$$|AD|\,|AB| + |DB|\,|AB| = |AC|^2 + |CB|^2$$
$$(|AD| + |AB|)\,|AB| = |AC|^2 + |CB|^2$$
$$|AB|\,|AB| - |AC|^2 + |CB|^2.$$

Set $|AC| = a$, $|CB| = b$ and $|AB| = c$, we have
$c^2 = a^2 + b^2$ which is the Pythagorean identity.

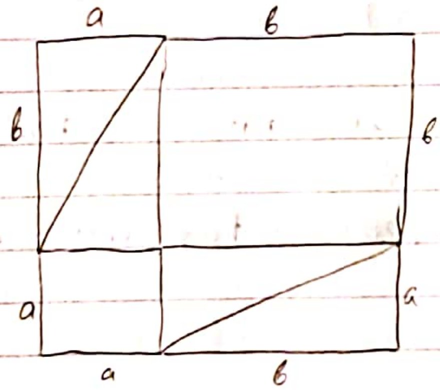
(2) Proof using squares.

Construct a square using 4 equal triangles the following way:



Then the area of this square is $(a+b)^2 = 4 \cdot \frac{1}{2}(a \cdot b) + c^2$
$= 2ab + c^2$

Now retransform these triangles the following way:



Here we can see that the area of this square is
$(a+b)^2 = 4 \cdot \frac{1}{2}(a \cdot b) + a^2 + b^2$
$= 2ab + a^2 + b^2$

Taking that from our 2 constructions:
$2ab + c^2 = (a+b)^2 = 2ab + a^2 + b^2$
we get that
$c^2 = a^2 + b^2$ which is the pythagorean theorem.

2. Prove that $\sqrt[7]{3}$ is irrational.

Proof:

Assume for contradiction that $\sqrt[7]{3}$ is rational, then we can write

$$\sqrt[7]{3} = \frac{m}{n}$$

where m, n are integers. WLOG take the smallest (m, n) pair. They will be coprime.

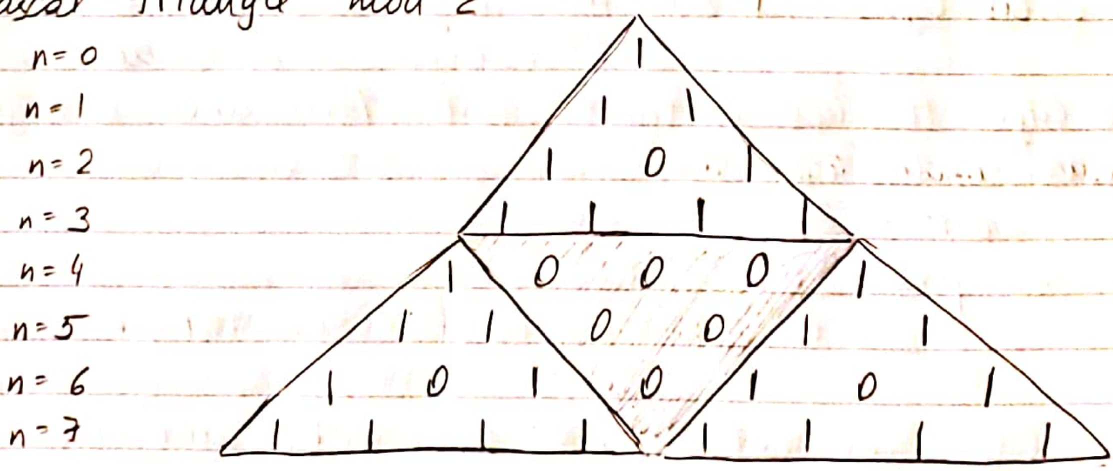
Now simplify by raising both sides to the power of 7:

$$3 = \frac{m^7}{n^7}$$

Then: $m^7 = 3n^7$. It follows that m^7 is a multiple of 3. We can also write the prime factorization of $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ and then $m^7 = p_1^{7a_1} p_2^{7a_2} \cdots p_k^{7a_k}$. Therefore m^7 must be a multiple of 3^7.

The same is true for n^7. But the right side is $3n^7$, therefore the power of 3 on the right is $7 \cdot a + 1$, while the power of 3 on the left is $7a$. This is a contradiction by the uniqueness of the prime decomposition

3. a) Pascal triangle mod 2



The shaded area is the center with all 0's, and the rest is made up of 3 identical triangles outlined each time.

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3 b) The Feigenbaum constant is $\delta = \lim_{n \to \infty} \frac{r_{k+1} - r_k}{r_k - r_{k-1}} \approx 4.6692$

Feigenbaum was interested in understanding populations modeled by $x_{n+1} = k x_n (1-x_n)$, where k is the reproduction constant. He noticed that
if $0 < k < 1$, population becomes extinct
if $1 < k < 3$ population stabilizes at some x.
if $3 < k < 3.5$ population will eventually alternate between 2 steady states.
Therefore, the steady state solutions will transition from 2^{k-1} to 2^k at some r_k. And he was able to determine that the differences between r_k's decrease each time by the Feigenbaum constant.

4. a) Platonic solid are regular polyhedrons in 3D. They are made up of identical regular polygons, with the same number of edges and faces meeting at every vertex. The relationship between edges E, vertices V, faces F is given by: $V - E + F = 2$.

b) Each edge touches 2 faces and there are b edges around each face so:

$b \cdot F = 2E$

$$\boxed{F = \frac{2E}{b}}$$

Each edge connects 2 vertexes (2E) and there are a edges at every vertex (a V) so:

$a \cdot V = 2E$

$$\boxed{V = \frac{2E}{a}}$$

4c) From Euler's formula: $V - E + F = 2$ and

$$E = \frac{b \cdot F}{2} \quad \text{and} \quad V = \frac{2E}{a} = \frac{bF}{a}$$

Plugging in:

$$\frac{bF}{a} - \frac{bF}{2} + F = 2$$

$$F\left(\frac{b}{a} - \frac{b}{2} + 1\right) = F\left(\frac{2b - ba + 2a}{2a}\right) = 2$$

$$\boxed{F = \frac{4a}{2b + 2a - ba}}$$

d) Possible Platonic figures:

(a,b)	F
$(3,3)$	$F = \frac{4 \cdot 3}{2\cdot3 + 2\cdot3 - 3\cdot3} = \frac{12}{3} = 4$
$(3,4)$	$F = \frac{4 \cdot 3}{2\cdot3 + 2\cdot4 - 3\cdot4} = \frac{12}{2} = 6$
$(3,5)$	$F = \frac{4 \cdot 3}{2\cdot3 + 2\cdot5 - 3\cdot5} = \frac{12}{16 - 15} = 12$
$(4,3)$	$F = \frac{4 \cdot 4}{2\cdot3 + 2\cdot4 - 4\cdot3} = \frac{16}{2} = 8$
$(4,4)$	$F = \frac{4 \cdot 4}{2\cdot4 + 2\cdot4 - 4\cdot4} = \frac{16}{0} = NA$ X
$(4,5)$	$F = \frac{4 \cdot 4}{2\cdot4 + 2\cdot5 - 4\cdot5} = \frac{16}{18 - 20} = -8$ X
$(5,3)$	$F = \frac{4 \cdot 5}{2\cdot3 + 2\cdot5 - 3\cdot5} = \frac{20}{16 - 15} = 20$
$(5,4)$	$F = \frac{4 \cdot 5}{2\cdot4 + 2\cdot5 - 4\cdot5} = \frac{20}{18 - 20} = -10$ X
$(5,5)$	$F = \frac{5 \cdot 5}{2\cdot5 + 2\cdot5 - 5\cdot5} = \frac{25}{-5} = -5$ X

Therefore the platonic solids are:

	F	V	E
(3, 3)	4	4	6
(3, 4)	6	8	12
(3, 5)	12	20	30
(4, 3)	8	6	12
(5, 3)	20	12	30

where $V = \frac{b \cdot F}{a}$, $E = \frac{b \cdot F}{2}$

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5. Proof of Lagrange's theorem:

If H is a subgroup of G, then we can do a left coset decomposition of G.
Pick some term $g_1 \in G$ s.t. $g_1 \notin H$. Then we have a left coset of G in H:

$$g_1 H = \{ g_1 h_1, g_2 h_2, \dots g_1 h_n \}.$$

We can continue choosing $g_i \in G$ s.t. $g_i \notin H$ or $g_i \notin g_j H$ for any $j < i$, to find unique cosets.
In the end we will have a coset decomposition of G:

$$G = g_1 H + g_2 H + g_3 H + \dots + g_k H.$$

Note that each of the left cosets is a subgroup and has the same cardinality as H because it was achieved by multiplying every element of H by a constant.
And note that the left cosets won't share any elements by construction (we proved this in a lemma in class).

Therefore, we need some integer number of left cosets to construct G. Therefore:

$$|G| = k \cdot |H| \quad \text{bc each coset has size } |H|$$

and $\frac{|G|}{|H|}$ is some integer.

6. The equation pair $u_x = v_y$; $u_y = -v_x$ is a Cauchy-Riemann equation that satisfies $u + iv = f(x + iy)$ which Riemann wrote about in his doctoral thesis on theory of complex functions.

7. Quaternions were discovered by William Rowan Hamilton who lived his entire life in Dublin.

8. Heron's formula is: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where A is the area of a triangle with sides a, b, c. Heron lived 1 century A.D.

9. Isaac Newton studied in Cambridge University, under Isaac Barrow. Barrow gave up his clerkship to Newton. After leaving Cambridge, Newton became master of the mint.

10. Leibnitz was born in Leipzig, but lived mostly in Hanover by invitation of King George I.

11. a) Viete's product: $\frac{2}{\pi} = \cos\frac{\pi}{4} \cos\frac{\pi}{8} \cos\frac{\pi}{16} \cos\frac{\pi}{32} \ldots$

b) John Neper (Naiper) and Henry Briggs initiated the use of logarithms.

12. a) Eulerian path: start at the vertex with an odd number of edges, go through all the edges exactly 1 time and finish at another vertex with odd number of edges.

b) For a Eulerian path to exist, there should be 2 vertices with an odd number of edges and all the other vertices should have an even number of edges.

c) The starting and ending vertices will have an uneven number of entries/exits. The starting vertex will have one lone entry, and the exit one lone exit edge. All the edges in between will come in pairs of exit/entry. Therefore, the vertices inside the path should have an even number of edges, because once it is entered from one edge, it must be exited through another.

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