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Email DrZlinear@gmail.com as soon as I tell you (around 4:20 pm)
Subject: mt1
with an attachment called
mt1FirstLast.pdf (e.g. mt1LehonardEuler.pdf)
**E-Mail address:** VC387 @ scarletmail rutgers. edu

**MATH 437 Exam I for Dr. Z.'s, Fall 2021 (Oct. 27, 2021)**

**No Calculators! No Cheatsheets! YOU MAY USE YOUR HISTORY NOTEBOOK (But not your Math Notebook)**
**Show your work! An answer without showing your work will get you zero points.**
**A random subset of the students will be picked for short (private!) chats via WebEx, in order to verify that there was no outside help**

Do not write below this line (office use only)

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1. (out of 10)

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total: (out of 120)

**1.** (10 pts.) Prove that there are infinitely many primes.

We will prove by contradiction. Suppose there only exist finitely many primes. We will call this finite list of primes $p_1, p_2, p_3, \dots, p_n$.
If we take the product of the primes and add 1, we will get the number $P = p_1 p_2 p_3 \cdots p_n + 1$. All of $p_1, p_2, p_3, \dots, p_n$ would not divide $P$ because you would get a remainder 1. This $P$ would be a new prime because $P > p_1, p_2, p_3, \dots, p_n$ and $p_1, p_2, p_3, \dots, p_n \nmid P$. This is a contradiction of our assuption that there exists a finite list of primes. Hence, there exist infinitely many primes.

**2.** (10 pts.) Prove that $\sqrt{29}$ is irrational.

Let us consider the infinite simple continued fraction:

$5 + \frac{1}{5 + \frac{1}{5 + \cdots}}$ we can represent this continued fraction with $\frac{a + \sqrt{N}}{b}$.

$x = 5 + \frac{1}{x} \Rightarrow x^2 - 5x - 1 = 0 \qquad x = \frac{+5 \pm \sqrt{25 + 4}}{2(1)} \rightarrow x = \frac{5 \pm \sqrt{29}}{2}.$

The infinite continued fraction can be represented as $\frac{5 + \sqrt{29}}{2}$

Since $x$ is an infinite continued fraction, it would be irrational because a rational number would have a finite continued fraction. This implies $\frac{5 + \sqrt{29}}{2}$ is irrational, which implies $\sqrt{29}$ is irrational as desired.

**3.** (10 pts) Derive (from scracth, only using geometric series, and calculus) the Taylor series around $x = 0$ of the function

$$\arctan x^3 \quad .$$

Explain all steps!

We know that $\arctan(x) = \int_0^x \frac{1}{1 - t^2}$ and the famous Taylor series $\frac{1}{1 - z} = \sum_{n=0}^{\infty} x^n$

By replacing $z$ with $-t^2$, we get

$$\frac{1}{1 + t^2} = \sum_{n=0}^{\infty} (-t^2)^n = \sum_{n=0}^{\infty} (-1)^n t^{2n}$$

Hence $\arctan x^3 = \int_0^{x^3} \frac{1}{1 + t^2} = \int_0^{x^3} \sum_{n=0}^{\infty} (-1)^n t^{2n} = \sum_{n=0}^{\infty} (-1)^n \int_0^{x^3} t^{2n}$

$$\arctan x^3 = \sum_{n=0}^{\infty} (-1)^n \left( \frac{t^{2n+1}}{2n+1} \bigg|_0^{x^3} \right) = \boxed{\sum_{n=0}^{\infty} (-1)^{n^3} \frac{x^{2n^3+1}}{2n^3+1}}$$

**4.** (10 pts. altogether) Prove that

$$\sum_{k=0}^{n} k(k-1)(k-2) = \frac{(n+1)n(n-1)(n-2)}{4} \quad ,$$

**(i)** (5 points): The Dr. Z. way (verifying it for sufficiently many special cases, explain how many you need)

The expression provided has a degree 4, so we would only need to verify 5 different values

$n=0$: $\sum_{k=0}^{0} k(k-1)(k-2) = 0 = \frac{0(0+1)(0-1)(0-2)}{4}$ ✓

$0=0$

$n=1$: $\sum_{k=0}^{n=1} k(k-1)(k-2) = 0+0 = \frac{1(1+1)(1-1)(1-2)}{4}$

$0=0$ ✓

$n=2$: $\sum_{k=0}^{2} k(k-1)(k-2) = 0+0+0 = \frac{2(2+1)(2-1)(2-2)}{4}$

$0=0$ ✓

$n=3$: $\sum_{0}^{3} k(k-1)(k-2) = 0+0+0+6 = \frac{3(3+1)(3-1)(3-2)}{4} = \frac{3(\cancel{4})(2)(1)}{\cancel{4}}$

$6=6$ ✓

$n=4$: $\sum_{0}^{4} k(k-1)(k-2) = 0+0+0+6+24 = \frac{\cancel{4}(4+1)(4-1)(4-2)}{\cancel{4}} = (5)(3)(2)$

$30=30$ ✓

**(ii)** (5 points): The traditional way, using **complete mathematical induction**.

Base case. $n=0$.

$$\sum_{k=0}^{0} k(k-1)(k-2) = \frac{(n+1)n(n-1)(n-2)}{4}$$

$$0(0-1)(0-2) = \frac{(0+1)(0)(0-1)(0-2)}{4}$$

$0=0$ ✓

Inductive case - suppose $n \to n-1$

$$\sum_{k=0}^{n-1} k(k-1)(k-2) = \frac{(n-1+1)(n-1)(n-1-1)(n-1-2)}{4} = \frac{(n)(n-1)(n-2)(n-3)}{4}$$

Inductive step:

$$\sum_{k=0}^{n} k(k-1)(k-2) = \left(\sum_{k=0}^{n-1} k(k-1)(k-2)\right) + \left(n(n-1)(n-2)\right)$$

$$\sum_{k=0}^{n} k(k-1)(k-2) = \frac{(n)(n-1)(n-2)(n-3)}{4} + n(n-1)(n-2) = \frac{(n)(n-1)(n-2)(n-3)}{4} + \frac{4n(n-1)(n-2)}{4}$$

$$\sum_{k=0}^{n} k(k-1)(k-2) = \frac{n(n-1)(n-2)(n-3) + 4n(n-1)(n-2)}{4} = \frac{(n+1)(n)(n-1)(n-2)}{4} \qquad \square$$

**5.** (10 points) Construct a seven by seven Magic Square.

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| 10 | 18 | 26 | 34 | 42 | 43 | 2 |
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| 28 | 29 | 37 | 45 | 4 | 12 | 20 |

**6.** (10 points) Arrange the following people according to their year-of-birth, from oldest to youngest.

~~Newton~~, ~~Archimedes~~, ~~Gallileo~~, ~~Euler~~, ~~Gauss~~, ~~Zeilberger~~, ~~Euclid~~, ~~Thales~~, ~~Brahmagupta~~, ~~Fibonacci.~~

For each person, state their century of birth.

Thales (625 BC), Euclid (300BC), Archimedes (287), Brahmagupta (590), Fibonnaci (1170)
Galileo (1564), Newton (1642), Euler (1707), Gauss (1777), Zéiberger (1950)

**7.** (10 points). What is an Egyptian fraction? Express $\frac{5}{6}$ as an Egyptian fraction

Egyptian fraction is the sum of fractions of the form $\frac{1}{n}$ with $n \in \mathbb{Z}$.

$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$.

**8.** (10 points) What is the difference between Ionian (Greek) mathematics and ancient Babylonian and Chinese mathematics? Who was the traditional father of Greek mathematics?

Euclid is the father of greek mathematics.

**9.** (10 points) What book, except for the bible, was the most reproduced and studied in the Western world? Who was its author?

Elements by Euclid.

**10.** (10 points) In a closed polyhedron, what is a relation between $V$, the number of vertices, $E$, the number of edges, and $F$, the number of faces? Who is it due to?

$V + F - E = 2$

This relation is due to Euler

**11.** (10 points) What is the name of the following constant:

$$\lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}-\log n\right) \quad .$$

What is its approximate value?

$\lim \left( \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n} - \log n \right) = 0.577216\ldots$

and is called constant of Euler.

**12.** (10 points) Using the beginning of the famous Taylor expansion, about $x = 0$ for $\sin x$, namely

$$\sin(x) = x - \frac{1}{6}x^3 + \ldots \quad ,$$

find the beginning (up to term $x^3$) of the Taylor series, about $x = 0$ of

$$f(x) = \sin\sin\sin x \quad ,$$

in the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x_3 + \ldots$$

**Ans.**:

$a_0 =$ 0 ; $a_1 =$ 1 ; $a_2 =$ 0 ; $a_3 =$ $-\frac{1}{6}$.