Sarah Magno 10/27/21

Exam 1

① For the sake of contradiction, suppose that there are only finitely many primes, call them $p_1, p_2, \dots, p_k$. We make a new number, call it P, such that

$$P = p_1 p_2 \cdots p_k + 1$$

Thus this number is either prime (which contradicts that there are finitely many primes, since we just found a new prime) or it is composite. So it must have a prime divisor. But it can not be divided by any of $p_1, p_2, \dots, p_k$ since they all leave remainder 1 after division. Thus P must have a new prime divisor that is not on the list, again contradicting that the list of primes is finite. Thus there are infinitely many primes.

② For the sake of contradiction, suppose that $\sqrt{29}$ is rational. Then we can express $\sqrt{29}$ as

$$\sqrt{29} = \frac{a}{b} \quad \text{where } a, b \text{ are positive integers}$$

Without loss of generality, we can assume that not both a and b are multiples of 29, because if they were, we can keep dividing out by 29 until at least one of them isn't.

Then, by squaring and transposing, we obtain

$$29 = \frac{a^2}{b^2}$$

$$a^2 = 29b^2$$

(2 con't) Thus a^2 is divisible by 29, so a is divisible by 29. This is a lemma that we will use in the proof. Thus a can be written as

$$a = 29m \quad \text{for some integer } m$$

Returning to the original equation, we see that

$$(29m)^2 = 29b^2$$

$$29^2 m^2 = 29b^2$$

$$29m^2 = b^2$$

$$b^2 = 29m^2$$

Thus b^2 is divisible by 29, so by the lemma, b is also divisible by 29. But this contradicts our assumption that both a and b are not multiples of 29. Thus $\sqrt{29}$ must be irrational.

③ We use the facts that $\arctan x = \int_0^x \frac{1}{1+t^2}$ and $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ to derive the Taylor series expansion of arctan x.

First, we make the replacement $z = -t^2$ to obtain

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-t^2)^n$$

which can be rewritten as

$$\sum_{n=0}^{\infty} (-1)^n t^{2n}$$

Then we use term-by-term integration to get

$$\arctan x = \int_0^x \frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n} = \sum_{n=0}^{\infty} (-1)^n \left[\frac{t^{2n+1}}{2n+1}\right]_0^x$$

(since we can pull $\sum_{n=0}^{\infty} (-1)^n$ outside of the integral)

$$\boxed{= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

Now that we derived the Taylor series of arctan, we can find $\arctan(x^3)$ by plugging x^3 into the formula to arrive at

$$\arctan(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{2n+1} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}}$$

as the Taylor series around $x = 0$.

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(4) (i.) Since both sides are degree 4 polynomials, this means that we need to check 5 cases. This is because if two polynomials of degree 4 meet at $4+1=5$ different places, then they are the same.

$n=0$: empty sum $= 0 = \frac{(0+1)(0)(0-1)(0-2)}{4} \Rightarrow 0=0$ ✓

$n=1$: $0+1(1-1)(1-2) = \frac{(1+1)(1)(1-1)(1-2)}{4} \Rightarrow 0=0$ ✓

$n=2$: $0+2(2-1)(2-2) = \frac{(2+1)(2)(2-1)(2-2)}{4} \Rightarrow 0=0$ ✓

$n=3$: $0+3(3-1)(3-2) = \frac{(3+1)(3)(3-1)(3-2)}{4} \Rightarrow 6=6$ ✓

$n=4$: $6+4(4-1)(4-2) = \frac{(4+1)(4)(4-1)(4-2)}{4} \Rightarrow 30=30$ ✓

(ii.) Base Case: $n=0$ $\quad 0 = \frac{(0+1)(0)(0-1)(0-2)}{4} \Rightarrow 0=0$ ✓

Inductive hypothesis: $\sum_{k=1}^{n-1} k(k-1)(k-2) = \frac{((n-1)+1)(n-1)((n-1)-1)((n-1)-2)}{4}$

$$= \frac{n(n-1)(n-2)(n-3)}{4}$$

Inductive step: $\sum_{k=1}^{n} k(k-1)(k-2) = \left[\sum_{k=1}^{n-1} k(k-1)(k-2)\right] + n(n-1)(n-2)$

by the inductive hypothesis

$$= \frac{n(n-1)(n-2)(n-3)}{4} + n(n-1)(n-2)$$

$$= \frac{n(n-1)(n-2)(n-3) + 4n(n-1)(n-2)}{4}$$

$$= \frac{n^4-5n^3+6n^2-n^3+5n^2-6n+4n^3-8n^2-4n^2+8n}{4}$$

$$= \frac{n^4-2n^3-n^2+2n}{4} = \frac{(n+1)n(n-1)(n-2)}{4}$$

						1						
					2		8					
				3		9		15				
			4	35	10	41	16	47	22			
		5	29	11	42	17	48	23	5	29		
	6		12	36	18	49	24	6	30		36	
7		13	37	19	43	25	7	31	13	37		43
	14		20	44	26	1	32	14	38		44	
		21	45	27	2	33	8	39	21	45		
			28	3	34	9	40	15	46			
				35		41		47				
					42		48					
						49						

final answer is 7x7 grid inside the box

(6) Thales (5th century BC), Euclid (4th century BC), Archimedes (3rd century BC), Brahmagupta (7th century), Fibonacci (12th century), Galileo (16th century), Newton (17th century), Euler (18th century), Gauss (19th century), Zeilberger (20th century)

(7) Fractions reduced to sums of unit fractions, which have 1 as the numerator

$\frac{1}{x} = \frac{6}{5} \quad \text{ceil}\left(\frac{1}{x}\right) = 2$

$\frac{5}{6} = \frac{1}{2} + \left(\frac{5}{6} - \frac{1}{2}\right) = \frac{1}{2} + \frac{2}{6} = \boxed{\frac{1}{2} + \frac{1}{3}}$

(8) Ionian (Greek) mathematics was pure mathematics. Ancient Babylonian and Chinese mathematics was based on practical computations. The traditional father of Greek mathematics was Thales.

(9) The book was called The Elements. Its author is Euclid.

(10) The relation is $V + F - E = 2$. It is due to Euler.

(11) Its name is "the constant of Euler," and its approximate value is 0.577216...

(12) We use the expansion for $\sin(x) = x - \frac{1}{6}x^3 + \dots$ and use composition of functions. Since

$$f(x) = \sin(\sin(\sin x))$$

We obtain

$$f(x) = \sin(\sin(x - \tfrac{1}{6}x^3 + \dots))$$

Now we plug in $(x - \frac{1}{6}x^3 + \dots)$ into the expansion of $\sin(w) = w - \frac{1}{6}w^3 + \dots$

$$f(x) = \sin\left(x - \frac{x^3}{6} - \frac{1}{6}\left(x - \frac{x^3}{6}\right)^3 + \dots\right) = \sin\left(x - \frac{x^3}{6} - \frac{1}{6}\left(x^3 - \frac{x^5}{2} + \frac{x^7}{12} - \frac{x^9}{216}\right) + \dots\right)$$

$$= \sin\left(x - \frac{x^3}{3} + \dots\right)$$

We repeat the process again by using the expansion of $\sin(z) = z - \frac{1}{6}z^3 + \dots$

$$f(x) = \left(x - \frac{x^3}{3} - \frac{1}{6}\left(x - \frac{x^3}{3}\right)^3 + \dots\right)$$

$$= x - \frac{x^3}{3} - \frac{1}{6}\left(x^3 - x^5 + \frac{x^7}{3} - \frac{x^9}{27} + \dots\right)$$

$$= x - \frac{x^3}{2} + \dots$$

Thus

$a_0 = 0$
$a_1 = 1$
$a_2 = 0$
$a_3 = -\frac{1}{2}$

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