Sarah Catovic Exam I 10/27/2021

1. Assume, to the contrary, that there are finitely many primes $p_1, p_2, \dots, p_k$. Let $n = p_1 p_2 \cdots p_k + 1$. When n is divided by any of the primes $p_1, \dots, p_k$ there is a remainder 1. So, either n is a prime itself, which would mean that there is a prime not on the list $p_1, \dots, p_k$ or n is divisible by a prime that is not $p_1, \dots, p_k$. In either case, there is a prime number greater than $p_1, \dots, p_k$ so the assumption that there are finitely many primes is false. Therefore, there must be infinitely many primes.

2. Assume, to the contrary, that $\sqrt{29}$ is rational. So, it can be written in the form $\frac{p}{q}$, with p, q in lowest terms. and $p, q \in \mathbb{Z}$

$\sqrt{29} = \frac{p}{q}$ so $29 = \frac{p^2}{q^2}$ so $29q^2 = p^2$

p^2 is divisible by 29, so p must be divisible by 29. So, $p = 29n$ for some $n \in \mathbb{Z}$.

$29q^2 = (29n)^2 \Rightarrow 29q^2 = 29n^2$

So q^2 is divisible by 29, thus q must be divisible by 29. So, p and q are both divisible by 29, but we assumed that p and q are in lowest terms. So, our assumption that $\sqrt{29}$ is rational must be false, therefore $\sqrt{29}$ is irrational.

3. We know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ and that $\int_0^x \frac{1}{1+t^2} = \arctan(x)$

$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$\arctan(x) = \int \frac{1}{1+x^2} = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

so $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

so therefore $\arctan(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}$

4. $\sum_{k=0}^{n} k(k-1)(k-2) = \frac{(n+1)n(n-1)(n-2)}{4}$

a) Since the sum is of degree 4, we need to show that it works for 5 cases.

$n=0: 0 = \frac{(1)(0)(-1)(-2)}{4} = 0$ Yes!

$n=1: (1)(0)(-1) = 0 = \frac{(2)(1)(0)(-1)}{4} = 0$ Yes!

$n=2: \quad 2(1)(0)=0 = \dfrac{(3)(2)(1)(0)}{4}$ Yes!

$n=3: \quad 3(2)(1)=6 = \dfrac{(4)(3)(2)(1)}{4} = \dfrac{24}{4} = 6$ Yes!

$n=4: \quad (4)(3)(2) = 24 = \dfrac{(5)(4)(3)(2)}{4} = \dfrac{120}{4} = 30$

$0+0+6+\cancel{0}+16$ ✓ $= 30$

Since it was proved for 5 cases, then it must be true for all cases.

b) Base case: $n=0: (0)(-1)(-2)=0 = \dfrac{(1)(0)(-1)(-2)}{4} = 0$ Yes!

If we assume it is true for $k \geq 0$, then it must be true for $k+1$

Inductive hypothesis: $0 + \dots + k(k-1)(k-2) = \dfrac{(n+1)n(n-1)(n-2)}{4}$ is assumed to be true

$0 + \dots + k(k-1)(k-2) + (k+1)(k)(k-1) = \dfrac{(n+2)(n+1)(n)(n-1)}{4}$

5.

1
2 _ 8
3 _ 9 _ 15
4 _ _ 10 _ 16 _ 22
5 _ 11 _ 17 _ 23 _ 29
6 _ 12 _ 18 _ 24 _ 30 _ 36
7 _ 13 _ 19 _ 25 _ 31 _ 37 _ 43
14 _ 20 _ 26 _ 32 _ 38 _ 44
21 _ 27 _ 33 _ 39 _ 45
28 _ 34 _ 40 _ 46
35 _ 41 _ 47
42 _ 48
49

$\dfrac{(n^2+1)n^2}{2}$

4	35	10	41	16	47	22
29	11	42	17	48	23	5
12	36	18	49	24	6	30
37	19	43	25	7	31	13
20	44	26	1	32	14	38
45	27	2	33	8	39	21
28	3	34	9	40	15	46

6. Brahmagupta (1000), Fibonacci (1100), Thales (1200), Archimedes (1300), Euclid (1400), Galileo (1500), Newton (1600), Euler (1700), Gauss (1800) Zeilberger (1900)

7. An Egyptian fraction is a sum of unit fractions, with 1 in the numerator. $\frac{5}{6}$ as an Egyptian fraction would be
$\frac{1}{5/6} = 6/5$ The ceiling of $6/5$ is 2 so it would be $\frac{1}{2} + \left(\frac{5}{6} - \frac{1}{2}\right)$
$\frac{1}{2} + \frac{2}{6}$, but $\frac{2}{6}$ simplifies to $\frac{1}{3}$ so the Egyptian fraction of $\frac{5}{6}$ is $\frac{1}{2} + \frac{1}{3}$.

8. Babylonian and Chinese mathematics was concerned with knowing math in order to use it, while Ionian or Greek mathematics was focused on understanding math not just using it. Thales was the father of Greek mathematics.

9. The Elements by Euclid

10. Aristotle $V = \frac{1}{2}(E) - 3F$

11. Machin constant the approximate value is π

12. $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1!} = x - \frac{1}{6}x^3 + \cdots$ $f(x) = \sum_{n=0}^{\infty} \frac{f^n(x)(x)^n}{n!}$

$\sin(\sin(\sin(x)))$

$\sin(\sin(x)) = \sum_{n=0}^{\infty} (-1)^n \frac{(\sin x)^{2n+1}}{2n+1!}$

$\sin(\sin(\sin(x))) = \sum_{n=0}^{\infty} (-1)^n \frac{(\sin(\sin(x)))^{2n+1}}{2n+1!}$

$f(x) = \sin(\sin(\sin(x)))$ $f'(x) = \cos(\sin(\sin x))(\cos(\sin x))(\cos(x))$

$f''(x) = -\sin(\sin(\sin x))\cos(\sin x)\cos(x) + \cos(\sin(\sin x)) - \sin(\sin(x))$