Nabhanya Neb Exam 1

(1) Prove that there are infinitely many primes.

Contradiction: Let's say there is a finite amount of primes

$p_1, p_2, \dots, p_k$

Take some value $p = p_1 \cdot p_2 \cdot \dots \cdot p_k + 1$

This number leaves a remainder of 1 when you divide by $p_1, \dots p_k$. Thus, it is either a prime, or is divisible by some prime larger than p_k. This is a conflict with our original statement. Therefore, there are infinitely many primes.

* (2) Prove that $\sqrt{29}$ is irrational.

contradiction:

$$\sqrt{29} = \frac{a}{b} \xrightarrow{\text{lowest values}} 29 = \frac{a^2}{b^2} \rightarrow 29b^2 = a$$

This means 29 divides by a^2. But 29 is prime,

$$29q = a \rightarrow 29b^2 = (29q)^2 = 29^2 q^2 \rightarrow b^2 = 29q^2$$

This means 29 divides by b^2 and b.
This is a contradiction, since $\sqrt{29} = \frac{a}{b}$ in lowest terms

(3) Taylor series around $x=0$ of $\arctan x^3$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

We learned that $\arctan y = \int_0^y \frac{1}{1+t^2}$ and $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

using $z = -t^2$ we get $\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-t^2)^n = \sum_{n=0}^{\infty} (-1)^n t^{2n}$

Now, $\arctan y = \int_0^y \sum_{n=0}^{\infty} (-1)^n t^{2n} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{t^{2n+1}}{2n+1} \right) \Big|_0^y$

setting $y = x^3$ we get $\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}}$

④ Prove that $\sum_{K=0}^{n} K(K-1)(K-2) = \frac{n(n+1)(n-1)(n-2)}{4}$

(i) Dr Z's method:

$n=0$: $0(0-1)(0-2) = \frac{0(0+1)(0-1)(0-2)}{4}$

$0 = 0$ ✓

$n=1$: $0 + 1(1-1)(1-2) = \frac{1(1+1)(1-1)(1-2)}{4}$

$0 = 0$ ✓

$n=2$: $0 + 0 + 2(2-1)(2-2) = \frac{2(2+1)(2-1)(2-2)}{4}$

$0 = 0$ ✓

$n=3$: $0 + 0 + 0 + 3(3-1)(3-2) = \frac{3(3+1)(3-1)(3-2)}{4}$

$6 = 6$ ✓

$n=4$: $0 + 0 + 0 + 6 + 4(4-1)(4-2) = \frac{4(4+1)(4-1)(4-2)}{4}$

$30 = 30$ ✓

Therefore, it holds for every n. $0 \leq n \leq 2$ was checked to ensure both sides $= 0$, since we multiply by n, $n-1$, or $n-2$. We checked $n=3$ and $n=4$ as well. This is enough because the degree is 4 and we checked 5 different values.

(ii) Induction:

base case $n=0$: $0(0-1)(0-2) = \frac{0(0+1)(0-1)(0-2)}{4}$

$0=0$ ✓

inductive step: assuming it is true for some $n-1$, we show it for n.

$$\frac{(n+1)(n+2)(n)(n-1)}{4} + n(n-1)(n-2) \stackrel{?}{=} \frac{n(n+1)(n-1)(n-2)}{4}$$

$$\frac{(n+1)(n+2)(n)(n-1) + 4[n(n-1)(n-2)]}{4} \stackrel{?}{=} \frac{n(n+1)(n-1)(n-2)}{4}$$

$$\frac{(n+1)(n+2)(n)(n-1) + 4(n(n-1)(n-2)]}{4} = \frac{n^4-2n^3-n^2+2n}{4}$$

$$\frac{n^4-2n^3-n^2+2n}{4} = \frac{n^4-2n^3-n^2+2n}{4}$$ ✓

⑤ 7×7 magic square:

1
2 — 8
3 — 9 — 15
4 — 10 — 16 — 22
5 — 11 — 17 — 23 — 29
6 — 12 — 18 — 24 — 30 — 36
7 — 13 — 19 — 25 — 31 — 37 — 43
14 — 20 — 26 — 32 — 38 — 44
21 — 27 — 33 — 39 — 45
28 — 34 — 40 — 46
35 — 41 — 47
42 — 48
49

Reflection for scaffold:

4	35	10	41	16	47	22
29	11	42	17	48	23	5
12	36	18	49	24	6	30
37	19	43	25	7	31	13
20	44	26	1	32	14	38
45	27	2	33	8	39	21
28	3	34	9	40	15	46

(6) Thales (6th century BC)
Euclid (4th century BC)
Archimedes (3rd century BC)
Brahmagupta (6th century)
Fibonacci (12th century)
Gallileo (14th century)
Newton (15th century)
Euler (18th century)
Gauss (18th century)
Zeilberger (20th century)

(7) A sum of fractions of form $\frac{1}{n}$ with n as any natural number.

$\frac{5}{6} = x$, $\frac{6}{5} = \frac{1}{1} \rightarrow \text{ceil}\left(\frac{6}{5}\right) = 2$

$\frac{1}{2} + \left(\frac{5}{6} - \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{3}$

(8) Ionian: conceptual / philosophy / abstract
Thales is father
Chinese: algebra / number theory
Babylonian similar to Chinese, practical

(9) "Elements" by Euclid

(10) $V + F - E = 2$ by Euler

(11) Euler's constant $\approx 0.577216\ldots$

(12)

(12) $f(x) = \sin\sin\sin x$

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

$f^0 = \sin\sin\sin x$

$f' =$