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10/27/21
Dr. Z

Midterm 1-437

(1.) Prove that there are infinitely many primes.

We will begin with a proof by contradiction. Suppose there are finitely many primes. Let's say there are n primes, from $p_1, p_2 \cdots p_n$. We call a new number $P = p_1 \cdot p_2 \cdots p_n + 1$ the product of all primes + 1. Now, dividing P by $p_1 \cdots p_n$ returns remainder 1. This means P is either a prime OR it is divisible by a prime larger than p_n. These are both contradictions because we have found a new larger prime than p_n. This process can be repeated infinitely, thus, infinite primes!

(2.) Prove $\sqrt{29}$ is irrational.

We can do this using Dr. Z's method or a proof by contradiction. We will use Dr. Z's method from class.

All rational numbers can be written as $\frac{a}{b}$ where a, b are two coprime integers.
$\frac{a}{b} = \text{quotient} + \frac{\text{remainder}}{b}$, with quotient $= q \in \mathbb{Z}$ and $r < b$.

This process can be continued over and over.

Applying this to our problem, we can say $\sqrt{29} = 1 + (\sqrt{29} - 1)$, where both sides are equal. Using our formula from above, $\frac{1}{\sqrt{29}-1} = \frac{\sqrt{29}+1}{(\sqrt{29}-1)(\sqrt{29}+1)} = \frac{\sqrt{29}+1}{28} = \frac{1}{28}(\sqrt{29}+1) = \frac{2}{28} + \frac{1}{28}(\sqrt{29}-1)$

We can repeat this process but we will go around in circles forever. Therefore, $\sqrt{29}$ is irrational.

*Note to Dr. Z, after completing the proof, I think it would have been better to maybe choose another method. But the proof still works.

(3.) Derive from scratch the Taylor Series around $x=0$ of $\arctan(x^3)$.

We will begin with our function $f(x) = \arctan(x^3)$
We know the derivative of $f(x) = \arctan(x) = \frac{1}{1+x^2}$. We will use this to calculate the first few derivatives.

So, $f'(x) = \frac{3x^2}{x^6+1}$

$$f''(x) = \frac{((6x)(x^6+1)) - (6x^5)(3x^2)}{(x^6+1)^2} = -\frac{12x^7 - 6x^6}{(x^6+1)^2}$$

$(x^6+1)^2 = x^{12} + 2x^6 + 1$

$$f'''(x) = \frac{(x^6+1)^2(84x^6 - 36x^5) - (12x^7 - 6x^6)(12x^{11} + 12x^5)}{(x^6+1)^{4}} = \frac{60x^{12} - 150x^6 + 1}{(x^6+1)^3}$$

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(3.) Plugging 0, we get:

$f(0) = \arctan(0^3) = 0$

$f'(0) = \frac{3(0)^2}{0^6+1} = 0$

$f''(0) = \frac{12(0)^7 - 6(0)}{(0^6+1)^2} = 0$

$f'''(0) = \frac{60(0)^{12} - 150(0)^6 + 1}{(0^6+1)^3} = \frac{1}{1} = 1$

Next Step: We know: $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, if we use our $f'(x)$ we can see that $f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$, so a simple swap shows us that $z = -x^2$ means that:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$ Keep this in mind

$$\text{So, } \arctan'(x) = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n}$$

Now, $\arctan(x) - \arctan(0) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, substituting x^3 in, we have:

$$\boxed{\arctan(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1} \rightarrow \text{we can say } \arctan(x^3) = x^3 - \frac{x^9}{3} + \frac{x^{15}}{5} - \dots}$$

(4) Prove $\sum_{k=0}^{n} k(k-1)(k-2) = \frac{(n+1)n(n-1)(n-2)}{4}$

Ran out of time :(

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(5.) Construct a 7x7 Magic Square.

$\frac{n(n^2+1)}{2} = \frac{7(50)}{2} = \frac{350}{2} = \underline{175}$ Sum of all rows

[7x7]: We will use the method from class in an easier way to follow.

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

Starting with 1 in the middle of the top row we move up and to the right each time.

We put the 1 in the middle, and similar to the

x
x x x
x x x x x

method, we will use a square for easier readability.

(6.) Age order. Oldest

Thales (7th Century BC)
Euclid (3rd/4th Century BC)
Archimedes (3rd Century BC)
Brahmagupta (6th Century)
Fibonacci (12th Century)
Galileo (16th Century)
Zeilberger (20th Century)
Youngest

(7.) An Egyptian Fraction is a finite sum of distinct fractions of the form $\frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \cdots$

$\frac{5}{6} = \frac{1}{2} + EF(\frac{2}{6}) = \boxed{\frac{1}{2} + \frac{1}{3}}$

(8) The Ionian math used logic for many of its problems, the Babylonians used base 60, the Chinese used base 2 AND base 10.

Archimedes is the traditional father of Greek Math.

(9.) Euclid's "Elements", thirteen books.

(10) $(V + F - E = 2)$, Euler is the discoverer!

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(11.) $\lim_{n\to\infty}(1+\frac{1}{2}+\dots\frac{1}{n}-\log n)$ is Euler's Constant, found by Euler.
It's approximation is 0.577216

(12) $\sin(x)=x-\frac{1}{6}x^3+\dots$

$f(x)=\sin(\sin(\sin(x)))$

Since $\sin(x)=x-\frac{1}{6}x^3+\frac{x}{5!}\dots$ then introducing another sin gives us,

$\sin(\sin(x)) = \frac{\sin(x)}{1}-\frac{(\sin^3x)^3}{6}+\dots$

So, $\sin(\sin(\sin(x)))$ follows the same pattern. But we want $a_0+a_1x+a_2x^2\dots$

So we derive; $f(x)=\sin(\sin(\sin(x)))$

$f'(x)=\cos(x)\cdot\cos(\sin(x))\cdot\cos(\sin(\sin(x)))$

$f''(x)=$

$f(0)+f'(0)x+\frac{1}{2!}f''(0)x^2 \longrightarrow f(0)=0+0+0$

$\sin(\sin(\sin(x)))=a_0+a_1x+a_2x^2 \Rightarrow$

$$\boxed{\begin{gathered}a_0=1\\a_1=0\\a_2=\frac{1}{6}\\a_3=0\\a_4=\frac{1}{6}\end{gathered}}$$