1. Suppose there are finitely many primes, $p_1, p_2 \ldots p_n$ where n is finite.
Then they are all divisible by $p_1 \times p_2 \times \ldots \times p_n$.
Therefore, $p_1 \times p_2 \times \ldots p_n + 1$ divided by any of them leaves remainder 1 - in other words, they're not divisible.
If $p_1 \times p_2 \times \ldots p_n + 1$ is composite, it must have a prime factorization containing prime(s) besides itself. Since it isn't divisible by any known primes there must exist another prime outside the known primes and the list is incomplete.
If $p_1 \times p_2 \times \ldots p_n + 1$ is prime then it itself is a prime outside of the known primes and the list is incomplete.
Either way, a finite list of primes is incomplete and so there are infinitely many primes.

2. Suppose $\sqrt{29}$ is rational. Then there must exist $p, q \in \mathbb{Z}$ where $q \neq 0$, $\gcd(p,q) = 1$ such that

$$\sqrt{29} = p/q$$

So

$$29 = p^2/q^2$$

$$29q^2 = p^2$$

Because 29 divides p^2 it must divide p. So for some $m \in \mathbb{Z}$,

$$p = 29m.$$

$$29q^2 = (29m)^2$$

$$\cancel{29}q^2 = 29m^2$$

By the same logic as before, 29 must divide q.
But then $\gcd(p,q) \neq 1$ - this is absurd, and so $\sqrt{29}$ must be irrational.

3. $\arctan(0) = 0$

$\arctan'(x^3) = \frac{3x^2}{1+x^6} = 0$

$\arctan''(x^3) = \frac{(1+x^6)(6x) - (6x^5)(3x^2)}{(1+x^6)^2}$

$= \frac{6x + 6x^7 - 18x^7}{1 + 2x^6 + x^{12}}$

$= 6\,\frac{x - 2x^7}{1 + 2x^6 + x^{12}} = 0$

$\arctan'''(x^3) = \frac{(1+2x^6+x^{12})(1-14x^6) - (12x^5+12x^{11})(x \cdot 2x^7)}{(1+2x^6+x^{12})^2}$

$\frac{1}{1-x} = \sum_0 x^n$

$\arctan'(x^3) = 3x^2\left(\frac{1}{1+x^6}\right) = 3x^2 \sum_{n=0}^{\infty} (-x^3)^n$

$= 3x^2 \sum_{n=0}^{\infty} (-1)^n (x^{3n})$

$\arctan(x^3) = 3\int x^2 \cdot \sum_{n=0}^{\infty} (-1)^n (x^{3n})$

$= x^3 ($

1) take derivative of arctan
2) substitute in for formula for $\frac{1}{1-x}$ with $x = -x^3$
3) integrate both sides to get $\arctan(x^3) = \ldots$
4) solve integral

4.i) formula to 4th value, sufficient to prove at 5 values ~~for~~ to prove for all values

$n=0: \quad 0(0-1)(0-2)=0 = \frac{(0+1)(0)(0-1)(0-2)}{4} = 0 \quad \checkmark$

$n=1: \quad 0+1(1-1)(1-2)=0 = \frac{(1+1)(1)(1-1)(1-2)}{4} = 0 \quad \checkmark$

$n=2: \quad 0+0+2(2-1)(2-2)=0 = \frac{(2+1)(2)(2-1)(2-2)}{4} = 0 \quad \checkmark$

$n=3: \quad 0+0+0+3(3-1)(3-2)=6 = \frac{(3+1)(3)(3-1)(3-2)}{4} = \frac{24}{4} = 6 \quad \checkmark$

$n=4: \quad 6+4(4-1)(4-2)=30 = \frac{(4+1)(\cancel{4})(4-1)(4-2)}{\cancel{4}} = 30 \quad \checkmark$

Because the formula is true at 5 values, it's true for all values.

ii) Checking formula at $n=0$:

$0(0-1)(0-2)=0 = \frac{(0+1)(0)(0-1)(0-2)}{4} = 0 \checkmark$

So the base case is correct.

Suppose the formula is true for $n=m$.

Then $\sum_{k=0}^{m} k(k-1)(k-2) = \frac{(m+1)(m)(m-1)(m-2)}{4}$

So: $\sum_{k=0}^{m+1} k(k-1)(k-2) = \frac{(m+1)(m)(m-1)(m-2)}{4} + (m+1)(\cancel{m+1-1})(\cancel{m+1-2})(m)(m-1)$

$= \frac{(m+1)(m)(m-1)(m-2)+4(m+1)(m)(m-1)}{4}$

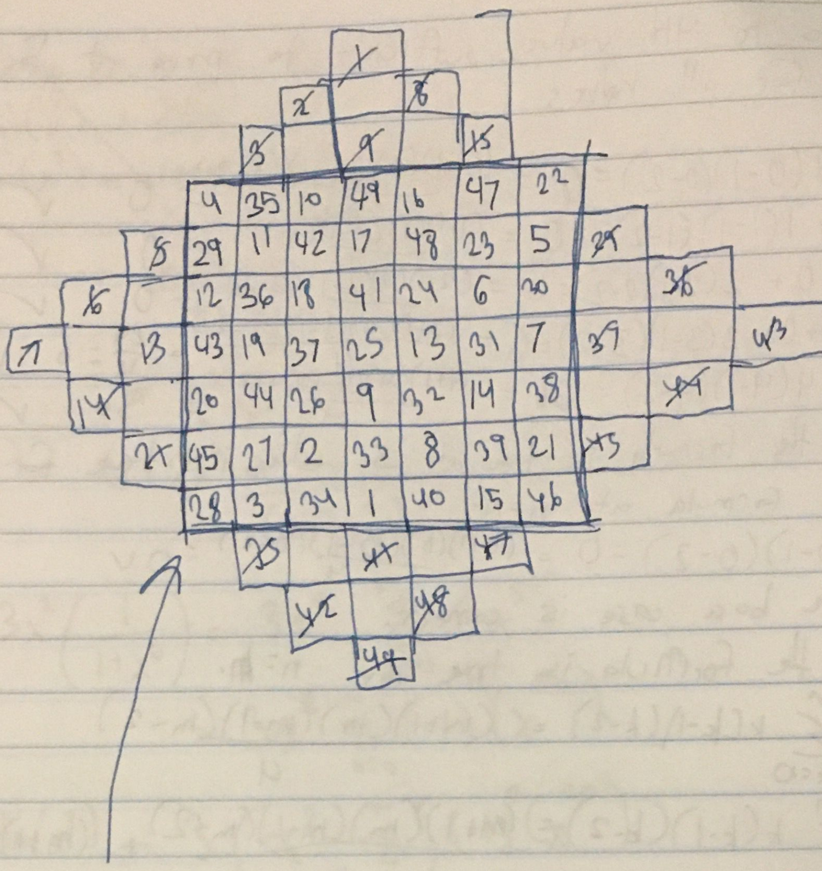
$= \frac{(m+1)(m-1)(m^2-2m+4m)}{4}$

$= \frac{(m+1)(m-1)(m)(m+2)}{4}$

$= \frac{((m+1)+1)(m+1)((m+1)-1)((m+1)-2)}{4}$

So the formula is true for $n=m+1$, and by induction is true for all values

5.



6. Thales
Euclid- 300's BC
Archimedes. 200's BC

Brahmagupta- 1100's
Fibonacci

Galileo - 1600's
Newton- 1600's

Gauss - 1700's
Euler

Zeilberger- 1900's

7. Egyptian fraction is fraction expressed as sum of unit fractions.

So $\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$

8. Ionian - geometry-based, emphasis on proof/demonstration
Babylonian + Chinese: algebra-based, no real proof/demonstration

~~Father~~ merchant Thales of Milete

9. Elements by Euclid

10.

11.

12. $\sin\sin\sin x = \sin\sin x - \cancel{\tfrac{1}{6}\sin^3\sin x}\ \frac{1}{6}\sin^3\sin x$

$\sin\sin x = \sin x - \frac{1}{6}\sin^3 x$

$\sin x = x - \frac{1}{6}x^3$

$\sin\sin x = x - \frac{1}{6}x^3 - \frac{1}{6}\left(x - \frac{1}{6}x^3\right)^3$

$= x - \frac{1}{6}x^3 - \frac{1}{6}\left(x^3 - \frac{1}{2}x^5 - \cancel{+}12x^7 \cancel{+} - 216x^9\right)$

$\approx x - \frac{2}{6}x^3$

$\sin\sin\sin x = x - \frac{2}{6}x^3 - \frac{1}{6}\left(x - \frac{2}{6}x^3\right)^3$

$= x - \frac{2}{6}x^3 - \frac{1}{6}\left(x^3 \ldots\right.$

$= x - \frac{1}{2}x^3$

$a_0 = 0,\ a_1 = 1,\ a_2 = 0,\ a_3 = -1/2$