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# Midterm

1. Prove that there are infinitely many primes.

<u>Proof:</u>

Assume the contrary - there are only a finite $n$ amount of primes and call them

$p_1, p_2, p_3 \dots p_n$, where $p_1 < p_2 < p_3 < \dots < p_n$

Now construct a number $P$ such that

$P = p_1 p_2 p_3 \cdot \ldots \cdot p_n + 1.$

Notice that any integer $N \in \mathbb{Z}$ can be factored into prime numbers. Therefore, we can do a prime factorization of $P$. We know that $P$ isn't divisible by any $p_1, p_2 \dots p_n$ because

$$\frac{P}{p_i} = \frac{p_1 p_2 \cdot \ldots \cdot p_i \cdot \ldots \cdot p_n}{p_i} + \frac{1}{p_i}$$

which won't be a whole number. So, it must follow that $P$ is divisible by some other prime numbers or $P$ is a prime number. In either case this shows that our list of prime numbers isn't exhaustive and we can always find a larger prime number. //

2. Prove that $\sqrt{29}$ is irrational.

Suppose that $\sqrt{29}$ is a rational number, then it can be written as

$$\sqrt{29} = \frac{p}{q}.$$

WLOG, take $p$ and $q$ to be relatively prime.

Now, square both sides:

$$29 = \frac{p^2}{q^2} \rightarrow p^2 = 29q^2.$$

We have that $p^2$ is divisible by 29, but since 29 is prime, $p$ must also be divisible by 29. So we can express $p = 29n$, for some $n \in \mathbb{Z}$.

Rewrite the equation with $p = 29n$ substitution:

$$p^2 = (29n)^2 = 29q^2$$

$$29^2 n^2 = 29q^2$$

$$29n^2 = q^2.$$

By the same logic as for $p$, we now showed that $q$ is divisible by 29 (because $q^2$ is divisible by 29).

This is a contradiction because we took $p$ and $q$ to be relatively prime.

Therefore $\sqrt{29}$ is irrational. //

3. Derive arctan $x^3$.

we know that $\arctan(x) = \int_0^x \frac{1}{t^2+1}\,dt$

and that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ so let

$x = -t^2$ and we will have that

$$\frac{1}{t^2+1} = 1 - t^2 + t^4 - t^6 + \dots$$

Now:

$$\arctan x^3 = \int_0^{x^3} \frac{1}{t^2+1}\,dt$$

$$= \int_0^{x^3} 1 - t^2 + t^4 - t^6 + \dots\,dt$$

$$= \left(t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \dots\right)\Big|_0^{x^3}$$

$$\underline{\arctan x^3 = x^3 - \frac{1}{3}x^9 + \frac{1}{5}x^{15} - \frac{1}{7}x^{21} + \dots}$$

$$\underline{\arctan x^3 = \sum (-1)^n \frac{x^{3(2n+1)}}{2n+1}}$$

4. Prove $\sum_{k=0}^{n} k(k-1)(k-2) = \frac{(n+1)n(n-1)(n-2)}{4}$

i) Dr Z's way:

The right hand side is a polynomial of degree 4, therefore need to show that it coincides with the left side 5 times to prove the equality:

$n=0 \qquad 0(0-1)(0-2) \stackrel{?}{=} \frac{(0+1)\cdot 0\cdot(0-1)(0-2)}{4}$

$0 = 0$ ✓

$n=1 \qquad 0+1(1-1)(1-2) \stackrel{?}{=} \frac{(1+1)\cdot 1\cdot(1-1)(1-2)}{4}$

$0 = 0$ ✓

$n=2 \qquad 0+0+2\cdot(2-1)(2-2) \stackrel{?}{=} \frac{(2+1)\cdot 2\cdot(2-1)(2-2)}{4}$

$0 = 0$ ✓

$n=3 \qquad 0+0+0+3\cdot(3-1)\cdot(3-2) \stackrel{?}{=} \frac{(3+1)\cdot 3(3-1)(3-2)}{4}$

$6 = 6$ ✓

$n=4 \qquad 0+0+0+6+4\cdot(4-1)(4-2) \stackrel{?}{=} \frac{(4+1)\cdot\cancel{4}(4-1)(4-2)}{\cancel{4}}$

$6+24 \stackrel{?}{=} 5\cdot 3\cdot 2$

$30 = 30$ ✓

Therefore, since the two expressions coincide at 5 terms, they must be equal.

ii) By Induction:

Take the base case when $n=0$, and show that $S_0$ holds:

$$\sum_{k=0}^{0} k(k-1)(k-2) \overset{?}{=} \frac{(0+1)\cdot 0(0-1)(0-2)}{4}$$

$$0 = 0 \quad \checkmark.$$

Therefore, $S_0$ holds. Now assume that $S_{n-1}$ holds and

$$\sum_{k=0}^{n-1} k(k-1)(k-2) = \frac{((n-1)+1)(n-1)((n-1)-1)((n-1)-2)}{4}$$

$$= \frac{n\cdot(n-1)(n-2)(n-3)}{4}.$$

Now show that $S_n$ holds:

$$\sum_{k=0}^{n} k(k-1)(k-2) = \left(\sum_{k=0}^{n-1} k(k-1)(k-2)\right) + n(n-1)(n-2)$$

$$= \frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)}{4} + n(n-1)(n-2)$$

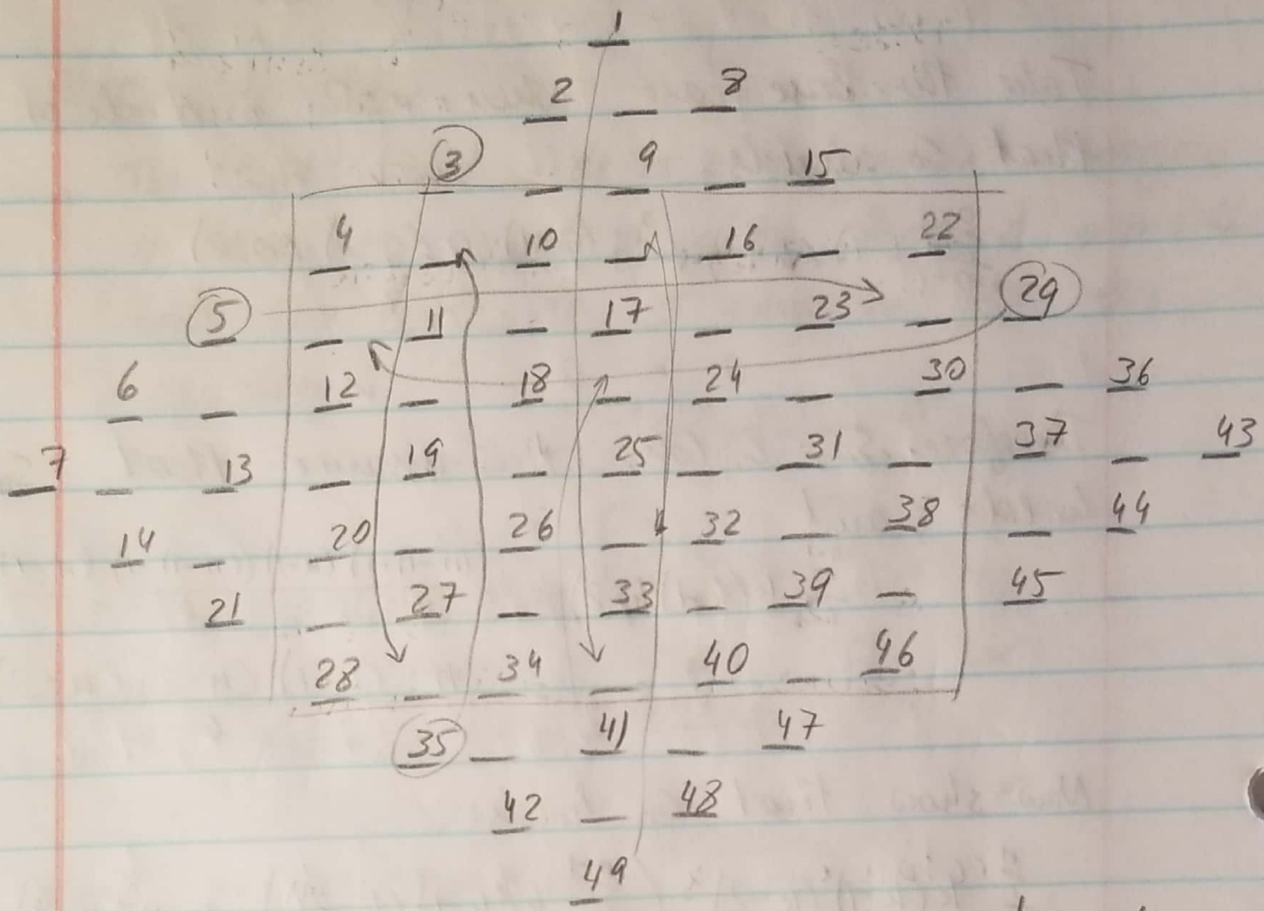
$$= \frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3) + 4n(n-1)(n-2)}{4}$$

$$= \frac{n(n-1)(n-2)(n-3+4)}{4}$$

$$= \frac{(n+1)\,n(n-1)(n-2)}{4}.$$

Therefore, $S_n$ holds so the equality is proven. //

5. Contruct a scaffold and fill diagonals:



Now reflect the numbers above the matrix down, below up, right to left and left to righ

The matrix is:

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | 35 | 10 | 49 | 16 | 47 | 22 |
| 29 | 11 | 42 | 17 | 48 | 23 | 5 |
| 12 | 36 | 18 | 41 | 24 | 6 | 30 |
| 43 | 19 | 37 | 25 | 7 | 31 | 13 |
| 20 | 44 | 26 | 9 | 32 | 14 | 38 |
| 45 | 21 | 27 | 33 | 8 | 39 | 21 |
| 28 | 3 | 34 | 1 | 40 | 15 | 46 |

6. People from oldest to youngest:

Thales - 7th BCE century
Euclid - 4th century BCE
Archimedes - 3rd century BCE
Brahmagupta - 6th century
Fibonacci - 12th century
Gallileo - 16th century
Newton - 17th century
Euler - 18th century
Gauss - 18th century
Zeilberger - 20th century.

7. An egyptian fraction is a fraction expressed as a sum of unit fractions.

$$\frac{5}{6} = \frac{1}{2} + \left(\frac{5}{6} - \frac{1}{2}\right) = \frac{1}{2} + \frac{2}{6}$$
$$= \boxed{\frac{1}{2} + \frac{1}{3}}$$

8. Thales of Milete was the father of Greek mathematics. Greek mathematics started answering the question 'why' something is done, a big step from ancient Babylonian and Chines math that cared only about algorithms and rules and not explanations.

9. 'Elements' by Euclid is the most reproduced and studied book in the western world.

10. Euler's polyhedron formula: $V - E + F = 2$.

11. $\lim_{n \to \infty} \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n \right)$ is the Euler-Mascheroni constant and it is equal to about 0.57721

12. $\sin(x) = x - \frac{1}{6} x^3 + \dots$
of $f(x) = \sin(\sin(\sin x))$ with center $x = 0$.
$f^0(0) = \sin(\sin(\sin 0)) = 0.$
$f'(0) = \cos(x) \cdot \cos(\sin x) \cdot \cos(\sin(\sin x)) = 1$
$f''(0) = 0$
$f'''(0) =$

so $a_0 = \sin(0) = 0$
$a_1 = \frac{1}{1!} = 1$
$a_2 = \frac{0}{2!} = 0$
$a_3 =$