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10/10/21

Homework for Lecture 8 - OK to post

① Suppose that there are two integers a and b, where $a > b > 0$, such that

$$\sqrt{2} = \frac{a}{b}$$

Then there exists a pair (a, b) where the sum, $a+b$, is as small as possible.

First, we prove an Intermediate Needed Fact: If m is an odd integer, then m^2 is odd.

Since m is odd, m can be written as $m = 2n+1$ for some integer n. Thus,

$m^2 = (2n+1)^2 = 4n^2 + 4n + 1 = 2(2n^2+2n)+1 = 2q+1$, where q is the integer $2n^2+2n$.

This shows that m^2 is odd, since it can be expressed in the form of an odd integer.

Next we use the Corollary that results from this: If m^2 is an even integer, then m is an even integer. We prove this by contradiction. Suppose m is an odd integer. Then the Intermediate Needed Fact tells us that m^2 is odd. This is a contradiction, since we assumed that m^2 was even. Thus m must be even.

Returning to the original equation $\sqrt{2} = \frac{a}{b}$, we square both sides and transpose to obtain

$$a^2 = 2b^2$$

Since $a^2 = 2b^2$, a^2 is even, and by the Intermediate Needed Fact, then a is even. Thus $a = 2m$ for some integer m, where $m = \frac{a}{2}$. Plugging in $2m$ for a, we see that

$$(2m)^2 = 2b^2$$
$$4m^2 = 2b^2$$
$$b^2 = 2m^2$$

This shows another pair of integers, where $b > m > 0$, such that

$$\sqrt{2} = \frac{b}{m}$$

But since $b + m = b + \frac{a}{2} < b + a$, this contradicts the assumption that $a > b > 0$ was the pair that had the smallest sum. We used the assumption that $\sqrt{2}$ was rational, and this led to a contradiction, so it is impossible for $\sqrt{2}$ to be rational. Thus $\sqrt{2}$ is irrational.

2. Using the geometrical representation, we see that

$$(2b-a)^2 = 2(a-b)^2$$

Rearranging this equation, we obtain the algebraic identity

$$a^2 - 2b^2 = -((2b-a)^2 - 2(a-b)^2)$$

Suppose that there are positive integers a, b where $a^2 - 2b^2 = 0$. Then there exists a pair such that $a+b$ is the smallest. Let

$$c = 2b - a \quad \text{and} \quad d = a - b$$

Plugging these values into the algebraic identity, we see that

$$c^2 - 2d^2 = 0$$

② But since $c+d = (2b-a)+(a-b) = b < a+b$, we have shown that the pair (c,d) also satisfies $\sqrt{2} = \frac{c}{d}$, but its sum is even smaller. This contradicts that (a,b) was the smallest pair. This assumption that the pair of integers (a,b) satisfies $a^2 = 2b^2$ led to the contradiction. Therefore, such a pair does not exist, so $\sqrt{2}$ must be irrational.

③ a.) $\frac{29}{16} = 1 + \frac{13}{16} = 1 + \frac{1}{\frac{16}{13}} = 1 + \frac{1}{1+\frac{3}{16}}$

b.) $\frac{32}{19} = 1 + \frac{13}{19} = 1 + \frac{1}{\frac{19}{13}} = 1 + \frac{1}{1+\frac{6}{13}}$

④ $\sqrt{2}+1 = 2 + (\sqrt{2}-1)$

$$= 2 + \frac{1}{\frac{1}{\sqrt{2}-1}}$$

We repeat the process on the denominator, which is $\frac{1}{\sqrt{2}-1}$.

$$\frac{1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}-1}\frac{(\sqrt{2}+1)}{(\sqrt{2}+1)} = \frac{\sqrt{2}+1}{2-1} = \sqrt{2}+1$$

We arrive at where we started, so we use self-similarity. We see that

$$\sqrt{2}+1 = 2 + \frac{1}{\sqrt{2}+1}$$

So if we let $x = \sqrt{2}+1$, we obtain

$$x = 2 + \frac{1}{x}$$

Thus $\sqrt{2}+1 = [2^{\infty}]$. This shows that $\sqrt{2}+1$ is irrational, since its continued fraction representation is infinite. This also shows that $\sqrt{2}$ is irrational, because since $\sqrt{2}+1$ is irrational, and we know that 1 is rational, then $\sqrt{2}$ is irrational, because the sum of an irrational number and a rational number is always irrational.

(5) $\frac{\sqrt{3}+1}{2} = 1 + \frac{\sqrt{3}-1}{2}$

$$= 1 + \cfrac{1}{\cfrac{1}{\frac{\sqrt{3}-1}{2}}}$$

We repeat this process on the denominator, which is $\frac{1}{\frac{\sqrt{3}-1}{2}}$.

$$\frac{1}{\frac{\sqrt{3}-1}{2}} = \frac{2}{\sqrt{3}-1}\frac{(\sqrt{3}+1)}{(\sqrt{3}+1)} = \frac{2(\sqrt{3}+1)}{3-1} = \frac{2(\sqrt{3}+1)}{2} = \sqrt{3}+1$$

Thus $\frac{\sqrt{3}+1}{2} = 1 + \frac{1}{\sqrt{3}+1}$

We repeat the process on the new denominator, which is $\sqrt{3}+1$.

$$\sqrt{3}+1 = 2 + (\sqrt{3}-1)$$

$$= 2 + \cfrac{1}{\cfrac{1}{\sqrt{3}-1}}$$

We repeat the process on $\frac{1}{\sqrt{3}-1}$.

$$\frac{1}{\sqrt{3}-1}\frac{(\sqrt{3}+1)}{(\sqrt{3}+1)} = \frac{\sqrt{3}+1}{3-1} = \frac{\sqrt{3}+1}{2}$$

We arrive at where we started, so by self-similarity, we see that

$$\frac{\sqrt{3}+1}{2} = 1 + \cfrac{1}{2 + \cfrac{1}{\frac{\sqrt{3}+1}{2}}}$$

If we let $x = \frac{\sqrt{3}+1}{2}$, we obtain

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{x}}$$

Thus $\frac{\sqrt{3}+1}{2} = [1, 2, 1, 2, 1, 2, \ldots]$. This shows that $\frac{\sqrt{3}+1}{2}$ is irrational, since its continued fraction representation is infinite.