Homework 8

(1) Every rational number, a can be written as
$\frac{a}{b} = q + \frac{r}{b}$. If $r=0$, we have an integer.

If not, $\frac{b}{r} > 1$ so $\frac{b}{r} = s + \frac{t}{r}$, etc, etc...

Now let's try with $\sqrt{2}$:

$$\sqrt{2} = 1 + \frac{(\sqrt{2}-1)}{1}$$

$$\frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{\sqrt{2}+1}{1} = \sqrt{2}+1 = 2 + (\sqrt{2}-1)$$

Now, we are back to $\sqrt{2}-1$, so it is clear that this process will never end, hence, $\sqrt{2}$ is not rational

(2) $a^2 - 2b^2 = -((2b-a)^2 - 2(a-b)^2)$
If there are positive integers a, b, such that $a^2 - 2b^2 = 0$, then there must be a pair for which $a+b$ is smallest

Let $c = 2b - a$ and $d = a - b$
$c+d = 2b - a + a - b = b < a+b$
So (c, d) also satisfies $\sqrt{2} = \frac{c}{d}$, but the sum is smaller.
This contradicts that (a, b) was smallest. Thus, the assumption that there exists (a, b) such that $a^2 = 2b^2$ is a contradiction. Hence, $\sqrt{2}$ is irrational

(3) (a) $\frac{29}{16} = 1 + \frac{13}{16} = 1 + \frac{1}{\frac{16}{13}} = 1 + \frac{1}{1 + \frac{3}{13}}$

(b) $\frac{32}{19} = 1 + \frac{13}{19} = 1 + \frac{1}{\frac{19}{13}} = 1 + \frac{1}{1 + \frac{13}{19}}$

(4) $\sqrt{2} + 1 = 1 + \frac{\sqrt{2}}{1} = 1 + \frac{1}{\frac{1}{\sqrt{2}}} = 1 + \frac{1}{1 + \left(\frac{1}{\sqrt{2}} - 1\right)} \cdots$

If we continue the process, we see that we get stuck in an infinite loop. Thus, $\sqrt{2}$ is irrational

(5) $\frac{\sqrt{3} + 1}{2} = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1}{2} + \frac{1}{\frac{2}{\sqrt{3}}} = \frac{1}{2} + \frac{1}{1 + \left(\frac{2}{\sqrt{3}} - 1\right)}$

$\ddots$