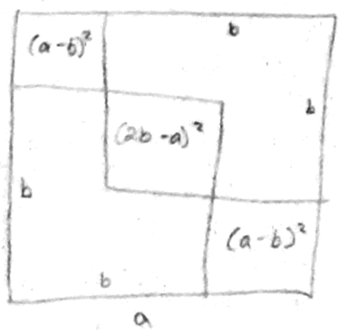
History of Math | Homework 8 | Karan Amin

(1) Assume $\sqrt{2}$ is rational, hence can be written as $\sqrt{2} = \frac{a}{b}$ for some integers a, b in smallest terms. Then we have that $2 = \frac{a^2}{b^2} \rightarrow 2b^2 = a^2$. Hence 2 divides a^2. Since 2 is prime, 2 divides a, hence we can write a as $a = 2k$ for some integer k. Therefore

$2b^2 = (2k)^2 = 4k^2 \rightarrow b^2 = 2k^2$, hence 2 divides b^2, therefore divides b. Similarly we can write b as $b = 2p$ for some $p \in \mathbb{Z}$. Then observe that a and b are both even, hence we can divide both by 2 to get a smaller fraction $\frac{k}{p}$ that represents $\sqrt{2}$ which is a contradiction. Hence $\sqrt{2}$ is irrational.

(2)



$(2b-a)^2 - 2(a-b)^2$

$4b^2 - 4ab + a^2 - 2(a^2 - 2ab + b^2)$

$= 4b^2 - \cancel{4ab} + a^2 - 2a^2 + \cancel{4ab} - 2b^2$

$= 2b^2 - a^2 = -(a^2 - 2b^2)$

Suppose there exists integers a, b such that $a^2 - 2b^2 = 0$. Let (a, b) be the pair with smallest sum

define

$c = 2b - a$, $d = a - b$

$c^2 - 2d^2 = 0$

but $c + d = 2b - a + a - b = b < a + b$ which is a contradiction

hence $\sqrt{2}$ is irrational

3)

a) $\frac{29}{16} = 1 + \frac{13}{16} = 1 + \frac{1}{\frac{16}{13}} = 1 + \frac{1}{1+\frac{3}{13}} = 1 + \frac{1}{1+\frac{1}{\frac{13}{3}}}$

$= 1 + \frac{1}{1+\frac{1}{4+\frac{1}{3}}}$

b) $\frac{32}{19} = 1 + \frac{13}{19} = 1 + \frac{1}{\frac{19}{13}} = 1 + \frac{1}{1+\frac{6}{13}} = 1 + \frac{1}{1+\frac{1}{\frac{13}{6}}} = 1 + \frac{1}{1+\frac{1}{2+\frac{1}{6}}}$

4) $\sqrt{2}+1 = 2 + (\sqrt{2}-1) = 2 + \frac{1}{\sqrt{2}+1} = 2 + \frac{1}{2+(\sqrt{2}-1)} = 2 + \frac{1}{2+\frac{1}{\sqrt{2}+1}}$

$= 2 + \frac{1}{2+\frac{1}{2+\frac{1}{\sqrt{2}+1}}}$ ...

Since it's an infinite continued fraction, it's irrational. Since 1 is rational, it must be that $\sqrt{2}$ is irrational.

5) $\frac{\sqrt{3}+1}{2} = 1 + \left(\frac{1}{2}\sqrt{3} - \frac{1}{2}\right) = 1 + \frac{1}{2+\left(\frac{1}{2}\sqrt{3} - \frac{5}{2}\right)}$

$= 1 + \frac{1}{2+\frac{1}{1+\left(\frac{1}{3}\sqrt{3} - \frac{7}{2}\right)}}$