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Homework 8

1. Traditional proof that $\sqrt{2}$ is irration:

Assume that $\sqrt{2}$ is a rational number. Then we can write $\sqrt{2} = \frac{m}{n}$, for $m, n \in \mathbb{Z}$.

Now take the pair of $(m,n)$ that give the smallest $m+n$. Then $(m,n)$ are coprime because if $m, n$ share a factor $z > 1 \in \mathbb{N}$, then $\frac{m}{n} = \frac{zm'}{zn'} = \frac{m'}{n'}$ and $m+n > m'+n'$.

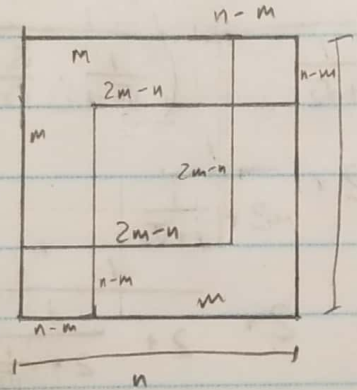
Now return to $\sqrt{2} = \frac{m}{n}$, s.t. $(m,n)$ are coprime. Then $2 = \frac{m^2}{n^2}$ and $m^2 = 2n^2$. From this it follows that $m^2$ is even, and $m$ is even by the Corollary: If the square of an integer is even then the integer is even:

Suppose $m^2$ is even, but $m$ is odd. Then can write $m = 2c+1$ for some $c \in \mathbb{Z}$ and $m^2 = (2c+1)^2 = (4c^2+4c)+1$, which is odd and gives a contradiction.

Therefore $m$ is even. Rewrite $m = 2a$ then $m^2 = (2a)^2 = 4a^2 = 2n^2$ and it follows that $n^2 = 2a^2$. By the same logic, since $n^2$ is even then $n$ must be even, and can be written as $n = 2b$ for some $b \in \mathbb{Z}$. That means that $n$ and $m$ share $2$ as a factor and aren't coprime. This is a contradiction. Therefore $\sqrt{2}$ is irrational. //

2. Geometric proof that $\sqrt{2}$ is irrational.

Take 2 squares, each with area $m^2$, such that they add up to the area of the square $n^2$. Then $2m^2 = n^2$ and $\sqrt{2} = \frac{n}{m}$. Take $n, m$ to be numbers that produce the smallest sum $n+m$. Then we can fit $m^2$ squares into $n^2$.




Then, since $2m^2 = n^2$, but $(2m-n)^2$ is overlapped, it must follow that the two $(n-m)^2$ squares make up for 1 $(2m-n)^2$ loss.

Therefore $2(n-m)^2 = (2m-n)^2$. But now we have constructed 2 squares of $(n-m)^2$ area that add up to $(2m-n)^2$ square:

$\sqrt{2} = \frac{2m-n}{n-m}$ and $(2m-n)+(n-m) = m < m+n$, which contradicts that $(m,n)$ were the smallest squares to exist (our assumption).

Thus it is impossible to build squares such that $2m^2 = n^2$, meaning that there are no integers s.t. $\sqrt{2} = \frac{m}{n}$. So, $\sqrt{2}$ is irrational.

3. a) $\frac{29}{16} = 1 + \frac{13}{16}$

$$= 1 + \frac{1}{\frac{16}{13}} = 1 + \frac{1}{1 + \frac{3}{13}}$$

$$= 1 + \frac{1}{1 + \frac{1}{13/3}} = 1 + \frac{1}{1 + \frac{1}{4 + 1/3}}$$

b) $\frac{32}{19} = 1 + \frac{13}{19}$

$$= 1 + \frac{1}{\frac{19}{13}} = 1 + \frac{1}{1 + \frac{6}{13}}$$

$$= 1 + \frac{1}{1 + \frac{1}{\frac{13}{6}}} = 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{6}}}$$

4. $\sqrt{2} + 1 = 2 + (\sqrt{2} - 1) = 2 + \frac{1}{1 + \sqrt{2}}$ $\left( \frac{1}{\frac{1}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1}} = \frac{1}{\sqrt{2}+1} \right)$

$$= 2 + \frac{1}{1 + 1 + (\sqrt{2} - 1)} = 2 + \frac{1}{2 + \frac{1}{1 + \sqrt{2}}}$$

$$= 2 + \frac{1}{2 + \frac{1}{1 + 1 + (\sqrt{2} - 1)}} = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\sqrt{2} + 1}}} \ldots$$

Therefore, this can go on forever and it shows that $(\sqrt{2} + 1)$ doesn't have a finite continued

5. $\frac{\sqrt{3}+1}{2} = 1 + \left( \frac{\sqrt{3}+1}{2} - 1 \right) = 1 + \frac{\sqrt{3} + 1 - 2}{2} = 1 + \frac{\sqrt{3} - 1}{2}$

$$= 1 + \frac{1}{\frac{2}{\sqrt{3} - 1}} = 1 + \frac{1}{\frac{2}{\sqrt{3}-1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1}} = 1 + \frac{1}{\frac{2\sqrt{3} + 2}{2}}$$

$$= 1 + \frac{1}{1 + \sqrt{3}} = 1 + \frac{1}{1 + 1 + (\sqrt{3} - 1)} = 1 + \frac{1}{2 + \frac{1}{\frac{\sqrt{3}+1}{2}}}$$

$$= 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\frac{\sqrt{3}+1}{2}}}}$$

Therefore, this fraction is infinite and $\frac{\sqrt{3}+1}{2}$ is irrational. Since 1 and 2 are rational it follows that $\sqrt{3}$ is irrational.

Chapter V

7. Italian mathematics of $16^{th}$ century

→ Scipio del Ferro discovered a new mathematics theory
- University of Bologna
  → students: Pacioli, Albrecht Dürer, Copernicus
- tried to find general solutions to

$$x^3+px=q, \quad x^3=px+q, \quad x^3+q=px$$

→ produced solutions to all three type
- died in 1526 without publishing

→ solutions rediscovered by a Venetian reckon master, Tartaglia
  → kept a secret, confidentially disclosed to doctor Cardano

→ Cardano published the solutions giving credit to Tartaglia (who didn't want them published)

'Ars Magna'

→ 'Quesiti' by Tartaglia vs 'Cartelli' by Ferrari
  → history of discovery to the public

- Cardano solution: for $x^3+px=q$

$$x = \sqrt[3]{\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}+\frac{q}{2}} - \sqrt[3]{\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}-\frac{q}{2}}$$

- Ars Magna + Ferrari's method of reducing the solution of the biquadratic equation to a cubic

Ferrari's equ: $x^4+6x^2+36=60x$

↳ $y^3+15y^2+36y=450$

→ negative numbers ↔ 'fictious'

- Raffael Bombelli - theory of imaginary complex numbers
  → solve irreducible case $\sqrt[3]{52+\sqrt{0-2209}} = 4+\sqrt{0-1}$
  → $3i = \sqrt{0-9}$

8. Platonic influence of Kepler's work and other parts of mathematics/astronomy

→ quantitative mathematical reasoning

→ 1593 - challenge by Adriaen Van Roomen to solve:

$x^{45} - 45x^{43} + 945x^{41} + \dots + 3795x^3 + 45x = A$

- special cases: $A = \sqrt{2 + \sqrt{2 + \sqrt{2 + 2}}}$ where

$x = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}}$

- ~~Alain~~ François Viete observed that the left part of the equation $= \sin \varphi$ when

solution: $\sin\left(\frac{\Phi}{45} - n.8^\circ\right)$

→ got rid of negative roots

Viete's main achievement - improvement of theory of equations

- represent numbers by letters
- 'logistica speciosa'
  - letters for numbers
  - signs +, -
  - 'A quadratum' $= A^2$
- still insisted on Greek homogeneity
  product of 2 line segments = area
- found $\pi$ in 9 decimals

$\frac{2}{\pi} = \cos\frac{\pi}{4} \cos\frac{\pi}{8} \cos\frac{\pi}{16} \cos\frac{\pi}{32} \dots$

8.

9. Simon Stevin, bookkeeper of Bruges
  → Engineer in the army of Prince Maurice of Orange
  - 'La disme' - decimal fractions to unify the whole
    system of measurement

John Neper, Scottish laird
  - 'Mirifici logarithmorum canonis descriptio'
  - two sequences related such that:
    one increase in arithmetic progression
    ⇓
    other decreases in a geometrical progression
  ⇒ product of two numbers of second sequence
  related to addition of the first
    - multiplication reduced to addition
  → $y = ae^{x/a}$ (or $x = $ Nep. $\log y$) where $a = 10^7$
    - clumsy attempt:
      $x = x_1 + x_2 \rightarrow y = \frac{y_1 y_2}{a}$ (instead of $y = y_1 y_2$)

Henry Briggs, professor at Gresham College in London.
  → $y = 10^x$ then $x = x_1 + x_2 \Rightarrow y = y_1 y_2$
  - published 'Arithmetica logarithmica'
    - 'Briggian' logarithms in 14 places for integers
      1 - 20.000 and 90.000 - 100.000

Ezechiel De Decker, a Dutch surveyor
  - 1627 - complete table of logarithms
    → filled in Briggs' gap 20.000 - 90.000