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Dr. Z, History of Math
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Homework for Lecture 4 - OK to post

① a.) At $t=0$,
A's distance from the start: 0
T's distance from the start: 50

It takes A $50=200t$, so $t=\frac{1}{4}$ hours to get to T's starting point,
but by then T went $50(\frac{1}{4})=12\frac{1}{2}$ extra miles, so

At $t=\frac{1}{4}$ hours,
A's distance from the start: 50
T's distance from the start: $62\frac{1}{2}$

It takes A $12\frac{1}{2}=200t$, so $t=\frac{1}{16}$ hours to get to where T was,
but by then T went $50(\frac{1}{16})=3\frac{1}{8}$ extra miles, so

At $t=\frac{1}{4}+\frac{1}{16}$ hours,
A's distance from the start: $62\frac{1}{2}$
T's distance from the start: $65\frac{5}{8}$

It takes A $3\frac{1}{8}=200t$, so $t=\frac{1}{64}$ hours to get to where T was, but
by then T went $50(\frac{1}{64})=\frac{50}{64}$ extra miles, so

At $t=\frac{1}{4}+\frac{1}{16}+\frac{1}{64}$ hours,
A's distance from the start: $65\frac{5}{8}$
T's distance from the start: $66\frac{13}{32}$

It takes A $\frac{50}{64}=200t$, so $t=\frac{1}{256}$ hours to get to where T was, but
by then, T went $50(\frac{1}{256})=\frac{25}{128}$ extra miles, so

① a.) At $t = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256}$ hours,
A's distance from the start: $66\frac{13}{32}$
T's distance from the start: $66\frac{77}{128}$

Zeno says that this process will continue forever, so A will never catch up to T.

b.) After t hours, A's distance from the start: $200t$ and T's distance from the start: $50+50t$. Equating these equations, we obtain

$$200t = 50 + 50t$$
$$150t = 50$$
$$t = \frac{1}{3}$$

So A and T meet after $\frac{1}{3}$ hours.

c.) We add up the intervals of time that A needed to catch up with T. We see that

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots = \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots\right) = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n$$

We evaluate using the formula $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, and we get

$$\frac{1}{4}\cdot\left(\frac{1}{1-\frac{1}{4}}\right) = \frac{1}{3}$$

So the answers in parts b and c are the same

① d.) We see that at $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}=\frac{21}{64}$ hours,

$$\text{dist}(A) = 65\tfrac{5}{8} = 65.625$$
$$\text{dist}(T) = 66\tfrac{13}{32} = 66.40625$$

At this point, we need $\frac{1}{256}$ seconds to continue, but this makes no sense, since the smallest unit of time is $\frac{1}{64}$ of an hour.

So at $t = \frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{64} = \frac{22}{64}$,

$$\text{dist}(A) = 65\tfrac{5}{8} + 200\left(\tfrac{1}{64}\right) = \frac{275}{4} = 68.75$$
$$\text{dist}(T) = 66\tfrac{13}{32} + 50\left(\tfrac{1}{64}\right) = \frac{1075}{16} = 67.1875$$

Since dist(A) > dist(T), A must have caught up to T at some point.

② For any integer $n \geq 0$ and any number x,

$$1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

We start by multiplying both sides by $(1-x)$ to obtain

$$(1-x)\left[1 + x + x^2 + \cdots + x^n\right] = \frac{1-x^{n+1}}{1-x}(1-x)$$

$$= 1 + \cancel{x} + \cancel{x^2} + \cancel{\cdots} + \cancel{x^n} \cancel{- x} \cancel{- x^2} \cancel{- x^3} - \cdots - x^{n+1} = 1 - x^{n+1}$$

$$1 - x^{n+1} = 1 - x^{n+1}$$

Since the left and right hand sides are equal, the statement holds true for any integer $n \geq 0$ and any number x.

③ a.) We start by writing the finite sum $\sum_{n=0}^{N} x^n$ and taking the limit as $N \to \infty$. We see that

$$\sum_{n=0}^{\infty} x^n = \lim_{N\to\infty} \sum_{n=0}^{N} x^N = \lim_{N\to\infty} \frac{1-x^{N+1}}{1-x} \quad \text{from Question \#2}$$

Since $0<x<1$, $\lim_{N\to\infty} x^{N+1} = 0$. Thus

$$\lim_{N\to\infty} \frac{1-x^{N+1}}{1-x} = \frac{1}{1-x}$$

b.) At $t=0$,
A's distance from the start: 0
T's distance from the start: 1

It takes A $1=1t$, so $t=1$ sec to get to T's starting point, but by then T went $x(1)=x$ extra miles, so

At $t=2$,
A's distance from the start: 1
T's distance from the start: $1+x$

It takes A $x=1t$, so $t=x$ secs to get to where T was, but by then T went $x(x)=x^2$ extra miles, so

At $t=3$,
A's distance from the start: $1+x$
T's distance from the start: $1+x+x^2$

It takes A $x^2=1t$, so $t=x^2$ secs to get to where T was, but by then T went $x(x^2)=x^3$ extra miles, so

③ b.) At $t=4$,

A's distance from the start: $1+x+x^2$

T's distance from the start: $1+x+x^2+x^3$

We see that the distance from the start follows the pattern of $1+x+x^2+\cdots+x^n$ for any integer $n \geq 0$, so the meeting time will occur if we write $1+x+x^2+\cdots x^n = \sum_{n=0}^{N} x^N$ and let $N \to \infty$, and we see that

$$\lim_{N\to\infty} \sum_{n=0}^{N} x^N = \lim_{N\to\infty} \frac{1-x^{N+1}}{1-x} = \frac{1}{1-x} \quad \text{since } x<1$$

Next we compute the meeting time the high school algebra way.

After t hours, A's distance from the start: $1t$ and T's distance from the start: $xt+1$

Equating these equations, we obtain

$$1t = xt+1$$

$$1t - xt = 1$$

$$t(1-x) = 1$$

$$t = \frac{1}{1-x}$$

So A and T meet after $\frac{1}{1-x}$ seconds.

④ We start by writing the partial sum of the first n terms. We let

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n}$$

Next we multiply both sides of this equation by 2 to obtain

$$2S_n = \frac{2}{2} + \frac{2}{4} + \frac{2}{8} + \frac{2}{16} + \dots + \frac{2}{2^n}$$

$$2S_n = 1 + \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{n-1}}\right]$$

$$2S_n = 1 + \left[S_n - \frac{1}{2^n}\right]$$

$$S_n = 1 - \frac{1}{2^n}$$

Finally, we let $n \to \infty$, and we see that $\frac{1}{2^n} \to 0$, so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

⑤

S	A	R	A	H
A	B	O	V	E
R	O	B	I	N
A	V	I	A	N
H	E	N	N	A