① $\sum_{k=1}^{n} 2k-1 = n^2$

a) Base $n=1 \Rightarrow$ $2-1=1$ ✓

Assume $p(n-1)$ is true which is that $\sum_{k=1}^{n-1} 2k-1 = (n-1)^2$

then $\sum_{k=1}^{n} 2k-1 = \sum_{k=1}^{n-1} 2k-1 + (2n-1) = (n-1)^2 + (2n-1)$

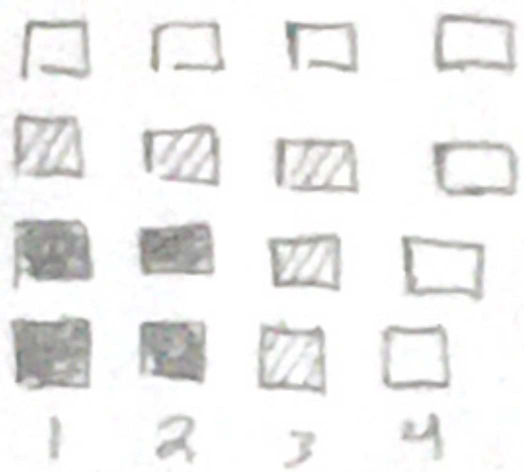
$= n^2 - 2n + 1 + 2n - 1 = n^2$

Hence $p(n) := \sum_{k=1}^{n} 2k-1 = n^2$ is true ∎

b) $n=1 \Rightarrow 2-1=1$

$n=2 \Rightarrow 1+4-1=4$

$n=3 \Rightarrow 4+6-1=9$



If you want to calculate for $n+1$, count the number of squares in th $n \times n$ block, then for $n+1$ we need $2n$ more blocks to construct the top and left row, the substract 1 because we have an extra corner block. Hence we now have $\sum_{k=1}^{n} 2k-1$. Geometrically this is a perfect square hence why its equal to n^2.

c) $n=0$ Empty Sum $= 0 = 0^2 = 0$ True

$n=1$ $2(1)-1 = 1 = 1^2 = 1$ True

$n=2$ $(2(1)-1) + (2(2)-1) = 1 + (4-1) = 4 = 2^2 = 4$ True

Since both sides are polynomials of degree two and they agree at three different values, the equation is true.

② $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

a) Base : $n=1$ $\quad 1 = \frac{1(2)}{2} = 1$

Assume $\sum_{k=1}^{n-1} k = \frac{(n-1)n}{2}$: $p(n-1)$ is true

Then $\sum_{k=1}^{n} k = \sum_{k=1}^{n-1} k + n = \frac{(n-1)n}{2} + n = \frac{n^2-n+2n}{2}$

$= \frac{n^2+n}{2} = \frac{n(n+1)}{2}$. Therefore $p(n)$ is true ∎

b) $T(n) = 1 + 2 + 3 + \cdots + n/2$

If n is even

$$\underbrace{\begin{array}{l} \quad + n \quad + n-1 + n-2 + \cdots + n/2+1 \\ \hline (n+1) + (n+1) + (n+1) + \cdots + (n+1) \end{array}}_{n/2 \text{ copies}}$$

$\Rightarrow$ Hence $\frac{n}{2} \cdot (n+1) = \frac{n(n+1)}{2}$

If n is odd $T(n) = 1 + 2 + 3 + \cdots + \frac{(n-1)}{2}$

$$\begin{array}{l} n + n-1 + n-2 + n-3 + \cdots + \frac{(n-1)}{2}+1 \\ \hline n + \underbrace{n + n + n + \cdots + + n}_{\frac{(n-1)}{2}} \end{array}$$

$\frac{(n-1)}{2} + \frac{(n-1)}{2} + 1 = n-1+1$
$= n$

$\frac{n-1}{2} \cdot n + n = \frac{n^2-n+2n}{2}$

$= \frac{n^2+n}{2} = \boxed{\frac{n(n+1)}{2}}$

c) $n=0$ $\quad$ Emptysum $= 0 = \frac{0 \cdot 1}{2} = 0$ True

$n=1$ $\quad 1 = \frac{1(2)}{2} = 1$ True

$n=2$ $\quad 1+2=3 = \frac{2(3)}{2} = 3$ True

Since both sides are polynomials of degree 2 and they agree at 3 different values, the equation is correct.

③ Derive an explicit formula for

$$\sum_{k=1}^{n} k^2 = A(n)$$

Summand is of degree 2

$A(n) = an^3 + bn^2 + cn + d$, a,b,c,d are to be determined

$A(0) = 0 = a\cdot 0^3 + b\cdot 0^2 + c\cdot 0 + d = d$

$A(1) = 1 = a\cdot 1^3 + b\cdot 1 + c\cdot 1 + d = a + b + c + d$

$A(2) = 1 + 2^2 = 5 = a\cdot 2^3 + b\cdot 2^2 + c\cdot 2 + d = 8a + 4b + 2c + d$

$A(3) = 5 + 3^2 = 5 + 9 = 14 = a\cdot 3^3 + b\cdot 3^2 + c\cdot 3 + d = 27a + 9b + 3c + d$

We know $d=0$, hence the new system is

$$\begin{aligned} a+b+c &= 1 \\ 8a+4b+2c &= 5 \\ 27a+9b+3c &= 14 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 5 \\ 27 & 9 & 3 & 14 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/6 \end{array}\right]$$

$a = 1/3$

$b = 1/2$

$c = 1/6$

$$A(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n^2+3n+1)}{6} = \frac{n(2n+1)(n+1)}{6}$$

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(2n+1)(n+1)}{6}}$$

a) Base $n=1$

LHS: $1^2 = 1$

RHS: $\frac{1(2+1)(2)}{6} = \frac{6}{6} = 1$

Assume $p(n-1)$ is true. Hence $\sum_{k=1}^{n-1} k^2 = \frac{(n-1)(2(n-1)+1)(n)}{6} = \frac{(n-1)n(2n-1)}{6}$ is true. Then $\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n-1} k^2 + n^2 = \frac{(n-1)n(2n-1) + 6n^2}{6} = \frac{n(2n^2 - n - 2n + 1) + 6n^2}{6}$

$$= \frac{2n^3 - n^2 - 2n^2 + n + 6n^2}{6} = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n+1)(n+1)}{6}$$

Hence $p(n)$ is true. ∎

b) $n=0$ Empty Sum $= 0 = \frac{0(0+1)(1)}{6} = 0$ True

$n=1$ $1^2 = 1 = \frac{1(2+1)(2)}{6} = \frac{6}{6} = 1$ True

$n=2$ $1 + 2^2 = 5 = \frac{2(4+1)(3)}{6} = \frac{30}{6} = 5$ True

$n=3$ $5 + 3^2 = 14 = \frac{3(7)(4)}{6} = \frac{84}{6} = 14$ True

Since both sides are polynomials of degree 3 and they agree on 4 different values, the equation is correct.

(4) Prove $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

Base case $n=1$

$$LHS = \sum_{k=1}^{1} k^3 = 1$$

$LHS = RHS$

$$RHS.\ \left(\frac{1(2)}{2}\right)^2 = 1$$

Assume $p(n-1)$ is true. That is that $\displaystyle\sum_{k=1}^{n-1} k = \left(\frac{(n-1)n}{2}\right)^2$

Then $$\sum_{k=1}^{n} k^3 = \sum_{k=1}^{n-1} k^3 + n^3 = \left(\frac{(n-1)n}{2}\right)^2 + n^3$$

$$= \frac{(n-1)^2 n^2 + 4n^3}{4} = \frac{(n^2-2n+1)n^2 + 4n^3}{4}$$

$$= \frac{n^4 - 2n^3 + n^2 + 4n^3}{4} = \frac{n^4 + 2n^3 + n^2}{4} = \frac{n^2(n^2+2n+1)}{4}$$

$$= \frac{n^2 \cdot (n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2$$

Hence $p(n)$ is true $\blacksquare$