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1. (a) Base case: when $n=1$, $1 = 1^2$ which is correct.

Induction step: $\sum_{k=1}^{n-1} 2k-1 = 1+3+5+7+\cdots+2(n-1)-1$

$= 1+3+5+7+\cdots+2n-3$

$= \frac{(1+2n-3)(n-1)}{2}$

$= \frac{(2n-2)(n-1)}{2}$

$= (n-1)^2$

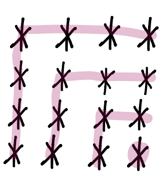
$\sum_{k=1}^{n} 2k-1 = (n-1)^2 + 2n-1$

$= n^2-2n+1+2n-1$

$= n^2$

QED

(b)



$\sum_{k=1}^{n} 2k-1 = n^2$

QED

(c) $n=1$, $2\times1-1 = 1^2$, true

$n=2$, $2\times1-1+2\times2-1 = 2^2$, true

$n=3$, $2\times1-1+2\times2-1+2\times3-1 = 3^2$, true

$n=4$, $2\times1-1+2\times2-1+2\times3-1+2\times4-1 = 4^2$, true

QED

2. (a)

Base case: when $n=1$, $1 = \frac{1\times2}{2} = 1$, which is correct

Induction step: $\sum_{k=1}^{n-1} k = 1+2+3+\cdots+n-1$

$= \frac{(1+n-1)(n-1)}{2}$

$= \frac{n(n-1)}{2}$

$$\sum_{k=1}^{n} k = 1+2+3+\cdots+n-1+n$$
$$= \frac{n(n-1)}{2} + n$$
$$= \frac{n(n-1)}{2} + \frac{2n}{2}$$
$$= \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

QED

(b) $\sum_{k=1}^{n} k = (1+n)+(2+n-1)+(3+n-2)+\cdots+\left(\frac{n}{2}+\frac{n}{2}+1\right)$, when n is even

Since there are $\frac{n}{2}$ terms.

$$\sum_{k=1}^{n} k = \frac{n}{2}(n+1)$$

Then $\sum_{k=1}^{n} k = (1+n)+(2+n-1)+(3+n-2)+\cdots+\left(\frac{n+1}{2}-1+\frac{n+1}{2}+1\right)+\frac{n+1}{2}$, when n is odd.

$$= \underbrace{(1+n)+(1+n)+(1+n)+\cdots+(1+n)}_{(\frac{n}{2}-1)\text{terms}} + \frac{n+1}{2}$$
$$= \left(\frac{n+1}{2}-1\right)(1+n) + \frac{n+1}{2}$$
$$= \frac{(n-1)(1+n)}{2} + \frac{n+1}{2}$$
$$= \frac{(n+1)n}{2}$$

QED

3. Since the summand is of degree 2, the quantity, let's call it $A(n)$

$$A(n) = \sum_{k=1}^{n} k^2 = an^3+bn^2+cn+d$$
$$A(0) = 0 = d$$
$$A(1) = 1 = a+b+c+d$$
$$A(2) = 1+2^2 = 5 = 8a+4b+2c+d$$
$$A(3) = 1+2^2+3^2 = 14 = 27a+9b+3c+d$$

$d=0, \begin{cases} a+b+c=1 \\ 8a+4b+2c=5 \\ 27a+9b+3c=14 \end{cases} \Rightarrow \begin{cases} 6a+2b=3 \\ 24a+6b=11 \end{cases} \Rightarrow \begin{cases} a=\frac{1}{3} \\ b=\frac{1}{2} \end{cases}$

$A(n)=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n = \frac{n(n+1)(2n+1)}{6}$

So $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

(a)

Base case: when $n=1$, $1^2 = \frac{1\times2\times3}{6}=1$, which is correct.

Induction step: $\sum_{k=1}^{n-1} k^2 = 1+2^2+3^2+\dots+(n-1)^2$

$= \frac{(n-1)n(2(n-1)+1)}{6}$

$= \frac{(n-1)n(2n-1)}{6}$

$\sum_{k=1}^{n} k^2 = \frac{(n-1)n(2n-1)}{6} + n^2$

$= \frac{(n-1)n(2n-1)+6n^2}{6}$

$= \frac{(n^2-n)(2n-1)+6n^2}{6}$

$= \frac{2n^3+3n^2+n}{6}$

$= \frac{n(n+1)(2n+1)}{6}$

(b) $n=0$: $0^2=0$, $0=0$, true

$n=1$: $1^2 = \frac{1\times2\times3}{6}=1$, $1=1$, true

$n=2$: $1^2+2^2 = \frac{2\times3\times5}{6}$, $5=5$, true

$n=3$: $1^2+2^2+3^2 = \frac{3\times4\times7}{6}$, $14=14$, true

QED

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

4. Proof.

Base case: when $n=1$, $1=\left(\frac{1\times2}{2}\right)^2=1$, which is correct.

Induction step: $\sum_{k=1}^{n-1} k^3 = \left(\frac{n(n-1)}{2}\right)^2$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n-1)}{2}\right)^2 + n^3$$

$$= \frac{n^2(n-1)^2 + 4n^3}{4}$$

$$= \frac{n^4 + 2n^3 + n^2}{4}$$

$$= \left(\frac{n(n+1)}{2}\right)^2$$