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Dr. Z, History of Math
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Homework for Lecture 3 - OK to post

① a.) Base Case: $n=0$ $\quad 0=0^2 \Rightarrow 0=0$ ✓

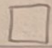
Inductive hypothesis: $\sum_{k=1}^{n-1} 2k-1 = (n-1)^2$

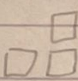
Inductive step: $\sum_{k=1}^{n} 2k-1 = \left(\sum_{k=1}^{n-1} 2k-1\right) + (2n-1)$

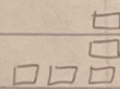
$= (n-1)^2 + (2n-1)$ by inductive hypothesis

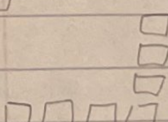
$= n^2 - 2n + 1 + 2n - 1$

$= n^2$

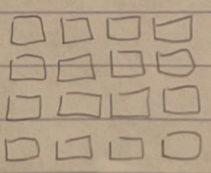
b.) The first odd number is 1, which can be represented as 

The second odd number is 3, which can be represented as 

The third odd number is 5, which can be represented as 

The fourth odd number is 7, which can be represented as 

Putting these shapes together, we obtain



which is a 4x4 square made up of the first four odd numbers, which shows that

$$\sum_{k=1}^{n} 2k-1 = n^2$$

① c.) $n=0$: $0 = 0^2$ ✓

$n=1$: $0 + (2(1)-1) = 1^2 \Rightarrow 1 = 1$ ✓

$n=2$: $0 + 1 + (2(2)-1) = 2^2 \Rightarrow 4 = 4$ ✓

Therefore it holds for every n, since the degree of both sides is 2 and we checked it in three different places.

② a.) Base case: $n=0$ $\quad 0 = \frac{0(0+1)}{2} \Rightarrow 0 = 0$ ✓

Inductive hypothesis: $\sum_{k=1}^{n-1} k = \frac{(n-1)((n-1)+1)}{2} = \frac{(n-1)n}{2}$

Inductive step: $\sum_{k=1}^{n} k = \left(\sum_{k=1}^{n-1} k\right) + n$

$$= \frac{n(n-1)}{2} + n \quad \text{by inductive hypothesis}$$

$$= \frac{n(n-1)+2n}{2}$$

$$= \frac{n^2+n}{2}$$

$$= \frac{n(n+1)}{2}$$

b.) Let S = the sum of the integers from 1 to n. Then

$$\begin{array}{r} 1+2+\cdots+n = S \\ n+(n-1)+\cdots+1 = S \\ \hline (n+1)+(n+1)+\cdots+(n+1) = 2S \end{array}$$

There are n copies of $(n+1)$ on the left hand side, so

$n(n+1) = 2S$ Solving for S, we obtain

$$S = \frac{n(n+1)}{2}$$

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② c.) $n=0: \quad 0=0$ ✓

$n=1: \quad 0+1 = \frac{1(1+1)}{2} \Rightarrow 1=1$ ✓

$n=2: \quad 0+1+2 = \frac{2(2+1)}{2} \Rightarrow 3=3$ ✓

Therefore it holds for every n, since the degree of both sides is 2 and we checked it in three different places.

③ Since k^2 is degree 2, then we know that $S(n) = \sum_{k=1}^{n} k^2$ will be a third degree polynomial. We let

$$S(n) = an^3 + bn^2 + cn + d$$

$S(0) = 0$
$S(1) = 0 + 1^2 = 1$
$S(2) = 1^2 + 2^2 = 5$
$S(3) = 1^2 + 2^2 + 3^2 = 14$

$S(0) = a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d = d$
$S(1) = a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d = a+b+c+d$
$S(2) = a \cdot 2^3 + b \cdot 2^2 + c \cdot 2 + d = 8a + 4b + 2c + d$
$S(3) = a \cdot 3^3 + b \cdot 3^2 + c \cdot 3 + d = 27a + 9b + 3c + d$

We see that

$$\left.\begin{array}{l} d=0 \\ a+b+c+d=1 \\ 8a+4b+2c+d=5 \\ 27a+9b+3c+d=14 \end{array}\right\} \begin{array}{l} a=\frac{1}{3} \\ b=\frac{1}{2} \\ c=\frac{1}{6} \\ d=0 \end{array}$$

So $S(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{2n^3+3n^2+n}{6}$

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③ a.) Base case: $n=0$ $\quad 0 = \frac{2(0)^3+3(0)^2+0}{6} \Rightarrow 0=0$ ✓

Inductive hypothesis: $\sum_{k=1}^{n-1} k^2 = \frac{2(n-1)^3+3(n-1)^2+(n-1)}{6}$

Inductive step: $\sum_{k=1}^{n} k^2 = \left(\sum_{k=1}^{n-1} k^2\right) + n^2$

$= \frac{2(n-1)^3+3(n-1)^2+(n-1)}{6} + n^2$ by inductive hypothesis

$= \frac{2(n^3-3n^2+3n-1)+3(n^2-2n+1)+(n-1)}{6} + n^2$

$= \frac{2n^3-6n^2+6n-2+3n^2-6n+3+n-1}{6} + n^2$

$= \frac{2n^3-3n^2+n}{6} + n^2$

$= \frac{2n^3-3n^2+n+6n^2}{6}$

$= \frac{2n^3+3n^2+n}{6}$

b.) $n=0$: $\quad 0 = \frac{2(0)^3+3(0)^2+0}{6} \Rightarrow 0=0$ ✓

$n=1$: $\quad 0+1^2 = \frac{2(1)^3+3(1)^2+1}{6} \Rightarrow 1=1$ ✓

$n=2$: $\quad 0+1^2+2^2 = \frac{2(2)^3+3(2)^2+2}{6} \Rightarrow 5=5$ ✓

$n=3$: $\quad 0+1^2+2^2+3^2 = \frac{2(3)^3+3(3)^2+3}{6} \Rightarrow 14=14$ ✓

Therefore it holds for every n, since the degree of both sides is 3 and we checked it in four different places.

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(4) $n=0: \quad 0^3 = \left(\frac{0(0+1)}{2}\right)^2 \Rightarrow 0=0$ ✓

$n=1: \quad 0^3+1^3 = \left(\frac{1(1+1)}{2}\right)^2 \Rightarrow 1=1$ ✓

$n=2: \quad 0^3+1^3+2^3 = \left(\frac{2(2+1)}{2}\right)^2 \Rightarrow 9=9$ ✓

$n=3: \quad 0^3+1^3+2^3+3^3 = \left(\frac{3(3+1)}{2}\right)^2 \Rightarrow 36=36$ ✓

$n=4: \quad 0^3+1^3+2^3+3^3+4^3 = \left(\frac{4(4+1)}{2}\right)^2 \Rightarrow 100=100$ ✓

Therefore it holds for every n, since the degree of both sides is 4 and we checked it on five different places.

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