Farrah Rahman
Can publish on site

1. (a) For all positive integers n, denote by $S(n)$ the statement that $\sum_{k=1}^{n} 2k-1 = n^2$. Then $S(1)$ states that:

$$\sum_{k=1}^{1} 2k - 1 = 1^2$$
$$2(1) - 1 = 1^2$$
$$1 = 1$$

So $S(1)$, the base case, is correct. Next, assume that for some arbitrary positive integer m, $S(m)$ holds true. Then $\sum_{k=1}^{m} 2k - 1 = m^2$. So,

$$\sum_{k=1}^{m} 2k - 1 = m^2$$
$$\sum_{k=1}^{m} 2k - 1 + 2(m+1) - 1 = m^2 + 2(m+1) - 1$$
$$\sum_{k=1}^{m+1} 2k - 1 = m^2 + 2(m+1) - 1$$
$$\sum_{k=1}^{m+1} 2k - 1 = m^2 + 2m + 1$$
$$\sum_{k=1}^{m+1} 2k - 1 = (m+1)^2$$

Thus $S(m+1)$ holds true. Because $S(m) \Rightarrow S(m+1)$, by the Principle of Mathematical Induction, $S(n)$ is true for all positive integers n.

(b) Consider the following 3x3 square:

```
. . .
. . .
. . .
```

The number of dots in the square, 9, can be split up into 3 L shapes:

```
________
.
________
. .
  .
________
. . .
    .
    .
________
```

Each of the l shapes has $2n - 1$ dots, where n is the number of dots in the width. So a square with a width of 3 is made of the dots $2(1) - 1 + 2(2) - 1 + 2(3) - 1$. This can be generalized into a formula for n dots, $\sum_{k=1}^{n} 2k - 1 = n^2$.

(c) Because the formula is of degree 2, proving it at three separate values is sufficient to prove it for all values.

$$
\begin{aligned}
S(1) \quad &: \quad \sum_{k=1}^{1} 2k - 1 = 1^2 \\
& 2(1) - 1 = 1^2 \\
& 1 = 1
\end{aligned}
$$

$$
\begin{aligned}
S(2) \quad &: \quad \sum_{k=1}^{2} 2k - 1 = 2^2 \\
& 2(1) - 1 + 2(2) - 1 = 4 \\
& 2 - 1 + 4 - 1 = 4 \\
& 4 = 4 \\
S(3) \quad &: \quad \sum_{k=1}^{3} 2k - 1 = 3^2 \\
& 2(1) - 1 + 2(2) - 1 + 2(3) - 1 = 9 \\
& 2 - 1 + 4 - 1 + 6 - 1 = 9 \\
& 9 = 9
\end{aligned}
$$

Thus the formula is correct.

2. (a) For all positive integers n, denote by $S(n)$ the statement that $\sum_{k=1}^{n} = \frac{(n)(n+1)}{2}$. Then $S(1)$ states that $\sum_{k=1}^{1} = \frac{(1)(1+1)}{2}$. This can be verified:

$$\begin{aligned}\sum_{k=1}^{1} &= \frac{(1)(1+1)}{2} \\ 1 &= \frac{(1)(2)}{2} \\ 1 &= 1\end{aligned}$$

So $S(1)$, or the base case, holds true. Next, for some arbitrary number k, assume that $S(m)$ is true. Then it's known that:

$$\begin{aligned}\sum_{k=1}^{m} &= \frac{(m)(m+1)}{2} \\ \sum_{k=1}^{m} +m+1 &= \frac{m^2+m}{2} + m + 1 \\ \sum_{k=1}^{m+1} &= \frac{m^2+m}{2} + \frac{2m+2}{2} \\ \sum_{k=1}^{m+1} &= \frac{m^2+m+2m+2}{2} \\ \sum_{k=1}^{m+1} &= \frac{m^2+3m+2}{2} \\ \sum_{k=1}^{m+1} &= \frac{(m+1)((m+1)+1)}{2}\end{aligned}$$

So $S(m+1)$ holds true. Because $S(m) \Rightarrow S(m+1)$, by the Principle of Mathematical Induction, $S(n)$ holds true for all positive integers n.

(b) Let n be an arbitrary integer.
First, assume n is even:

$$\begin{aligned}\sum_{k=1}^{n} &= 1+2+3+\cdots n-2+n-1+n \\ &= (1+(n-2))+(2+(n-1))+(3+(n-2))+\cdots (n/2)+(n/2+1)) \\ &= (n+1)+(n+1)+(n+1)+\cdots (n+1)\end{aligned}$$

Each of these $(n+1)$ are the sum of two distinct terms from the set $\{1, 2, \cdots n\}$. Therefore, the above sum is made up of $n/2$ $(n+1)$'s. In other words, when n is even, $\sum_{k=1}^{n} = \frac{(n)(n+1)}{2}$.
Next, assume that n is odd:

$$\begin{aligned}\sum_{k=1}^{n} &= 1+2+3+\cdots n-2+n-1+n \\ &= (1+(n-2))+(2+(n-1))+(3+(n-2))+\cdots ((n/2-1/2)+(n/2-1/2+2))+n/2+1/2 \\ &= (n+1)+(n+1)+(n+1)+\cdots (n+1)+\frac{n+1}{2}\end{aligned}$$

The $(n+1)$ terms are each formed from distinct terms of the numbers from 1 to n, excluding the middle number, $n/2+1/2$. Therefore there are $\frac{n-1}{2}$ groups of $(n+1)$'s, which sum to $\frac{(n-1)(n+1)}{2}$. The middle number can be added to this to get the sum of the numbers from 1 to n:

$$\begin{aligned}\frac{(n-1)(n+1)}{2}+\frac{n+1}{2} &= \frac{(n-1)(n+1)+(n+1)}{2} \\ &= \frac{(n+1)(n-1+1)}{2} \\ &= \frac{(n+1)(n)}{2}\end{aligned}$$

Thus when n is odd, the sum of the numbers from 1 to n is $\frac{(n+1)(n)}{2}$. Because the same formula holds for when n is even or odd, for all positive integers n, the sum of the numbers from 1 to n is $\frac{(n+1)(n)}{2}$.

(c) Because the formula is of the second power, proving it for three values is sufficient to prove it for all values.

$$\begin{aligned} S(1) \quad &: \quad \sum_{k=1}^{1} k = \frac{(1)(1+1)}{2} \\ & \quad\quad 1 = 2/2 \\ & \quad\quad 1 = 1 \\ S(2) \quad &: \quad \sum_{k=1}^{2} k = \frac{(2)(2+1)}{2} \\ & \quad\quad 1+2 = \frac{(2)(3)}{2} \\ & \quad\quad 3 = 6/2 \\ & \quad\quad 3 = 3 \\ S(3) \quad &: \quad \sum_{k=1}^{3} k = \frac{(3)(3+1)}{2} \\ & \quad\quad 1+2+3 = \frac{(3)(4)}{2} \\ & \quad\quad 6 = 12/2 \\ & \quad\quad 6 = 6 \end{aligned}$$

Thus the formula is correct.

3. Because the summand is of the second degree, the total sum must be of the third degree. So it must follow the form $an^3 + bn^2 + cn + d$. Four variables can be solved through four equations:

$$\begin{aligned}
1^2 &= a(1^3) + b(1^2) + c(1) + d \\
1^2 + 2^2 &= a(2^3) + b(2^2) + c(2) + d \\
1^2 + 2^2 + 3^2 &= a(3^3) + b(3^2) + c(3) + d \\
1^2 + 2^2 + 3^2 + 4^2 &= a(4^3) + b(4^2) + c(4) + d
\end{aligned}$$

These can be simplified:

$$\begin{aligned}
1 &= a + b + c + d \\
5 &= 8a + 4b + 2c + d \\
14 &= 27a + 9b + 3c + d \\
30 &= 64a + 16b + 4c + d
\end{aligned}$$

This eventually gets the system $a = 1/3$, $b = 1/2$, $c = 1/6$, $d = 0$. This results in the polynomial $\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{2n^3+3n^2+n}{6}$.

(a) This can be proved through induction:
For all natural positive integers n, denote by $S(n)$ the statement that $\sum_{k=1}^{n} k^2 = \frac{2n^3+3n^2+n}{6}$. Then $S(1)$ states that $\sum_{k=1}^{1} k^2 = \frac{2*1^3+3*1^2+1}{6}$. This can be verified:

$$\begin{aligned}
\sum_{k=1}^{1} k^2 &= \frac{2 * 1^3 + 3 * 1^2 + 1}{6} \\
1^2 &= \frac{2 + 3 + 1}{6} \\
1 &= \frac{6}{6} \\
1 &= 1
\end{aligned}$$

Thus $S(1)$, or the base case, is correct. Next, assume for some arbitrary positive integer m that $S(m)$ is correct. Then,

$$\begin{aligned}
\sum_{k=1}^{m} k^2 &= \frac{2m^3 + 3m^2 + m}{6} \\
\sum_{k=1}^{m} k^2 + (m+1)^2 &= \frac{2m^3 + 3m^2 + m}{6} + (m+1)^2 \\
\sum_{k=1}^{m+1} k^2 &= \frac{2m^3 + 3m^2 + m + 6(m+1)^2}{6} \\
&= \frac{2m^3 + 3m^2 + m + 6m^2 + 12m + 6}{6} \\
&= \frac{(2m^3 + 6m^2 + 6m + 2) + (3m^2 + 6m + 3) + m + 1}{6} \\
&= \frac{2(m+1)^3 + 3(m+1)^2 + m + 1}{6}
\end{aligned}$$

Thus $S(m+1)$ is correct. Because $S(m) \Rightarrow S(m+1)$, by the Principle of Mathematical Induction, $S(n)$ is correct for all positive integers n.

(b) Because the formula is of order 3, it can be proved by verifying 4 values.

$$
\begin{aligned}
S(1) \quad &: \quad \sum_{k=1}^{1} k^2 = \frac{2(1^3)+3(1^2)+1}{6} \\
& 1 = \frac{2+3+1}{6} \\
& 1 = \frac{6}{6} \\
& 1 = 1 \\
S(2) \quad &: \quad \sum_{k=1}^{2} k^2 = \frac{2(2^3)+3(2^2)+2}{6} \\
& 1+2^2 = \frac{2(8)+3(4)+1}{6} \\
& 5 = \frac{16+12+2}{6} \\
& 5 = \frac{30}{6} \\
& 5 = 5 \\
S(3) \quad &: \quad \sum_{k=1}^{3} k^2 = \frac{2(3^3)+3(3^2)+3}{6} \\
& 1+2^2+3^2 = \frac{2(27)+3(9)+3}{6} \\
& 14 = \frac{54+27+3}{6} \\
& 14 = \frac{84}{6} \\
& 14 = 14 \\
S(4) \quad &: \quad \sum_{k=1}^{4} k^2 = \frac{2(4^3)+3(4^2)+4}{6} \\
& 1+2^2+3^2+4^2 = \frac{2(64)+3(16)+4}{6} \\
& 30 = \frac{128+48+4}{6} \\
& 30 = \frac{180}{6} \\
& 30 = 30
\end{aligned}
$$

Thus the formula is correct.

4. For all positive integers n, denote by $S(n)$ the statement that $\sum_{k=1}^{n} k^3 = (\frac{(n)(n+1)}{2})^2$. Then $S(1)$ states that:

$$1^3 = \left(\frac{(1)(2)}{2}\right)^2$$
$$1 = 1$$

Thus $S(1)$, or the base case, is correct. Next assume for some arbitrary positive integer m that $S(m)$ is correct. Then:

$$1^3 + 2^3 + \cdots + m^3 = \left(\frac{(m)(m+1)}{2}\right)^2$$
$$1^3 + 2^3 + \cdots + m^3 + (m+1)^3 = \left(\frac{(m)(m+1)}{2}\right)^2 + (m+1)^3$$
$$1^3 + 2^3 + \cdots + m^3 + (m+1)^3 = \left(\frac{(m^2+m)}{2}\right)^2 + m^3 + 3m^2 + 3m + 1$$
$$1^3 + 2^3 + \cdots + m^3 + (m+1)^3 = \frac{m^4 + 2m^3 + m^2 + 4m^3 + 12m^2 + 12m + 4}{4}$$
$$1^3 + 2^3 + \cdots + m^3 + (m+1)^3 = \frac{(m+2)^2(m+1)^2}{4}$$
$$1^3 + 2^3 + \cdots + m^3 + (m+1)^3 = \left(\frac{(m+1)(m+2)}{2}\right)^2$$

So $S(m+1)$ is true. Because $S(m) \Rightarrow S(m+1)$, by the Principle of Mathematical Induction, $S(n)$ is true for all positive integers n.