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HW3.

1. Prove $\sum_{k=1}^{n} 2k-1 = n^2$

a) Take base case $n=1$:

$$(2-1) \stackrel{?}{=} 1^2$$

$$1 = 1 \checkmark$$

Now assume this equality holds for $(n-1)$

$$\sum_{k=1}^{n-1} 2k-1 = (2-1)+(2\cdot 2-1)+\ldots+(2(n-1)-1) = (n-1)^2.$$

Now consider case n:

$$\sum_{k=1}^{n} 2k-1 = \left[(2-1)+(2\cdot 2-1)+\ldots+(2(n-1)-1)\right]+(2n-1) \stackrel{?}{=} n^2$$

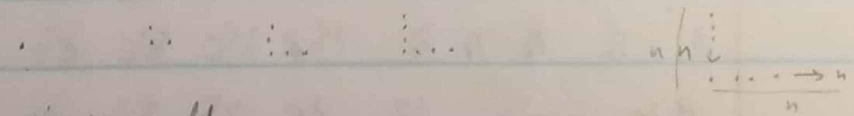
$$= \left(\sum_{k=1}^{n-1} 2k-1\right)+(2n-1)$$

$$= (n-1)^2+(2n-1)$$

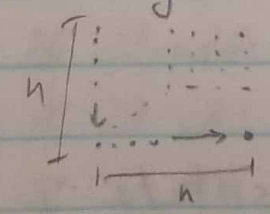
$$= n^2-2n+1+2n-1$$

$$= n^2. \quad \text{Equality holds.}$$

b) $\sum_{k=1}^{n} 2k-1 = \overset{1}{1} + \overset{2}{3} + \overset{3}{5} + \overset{4}{7} + \ldots + 2n-1$



Then summing all



$\rightsquigarrow n \times n$

c) $\sum_{k=1}^{n} 2k-1 = n^2 \rightarrow$ of degree 2. So need to prove to $n=3$.

$n=1$: $2\cdot 1-1 \stackrel{?}{=} 1^2 \rightarrow 1=1 \checkmark$

$n=2$: $(2\cdot 1-1)+(2\cdot 2-1) \stackrel{?}{=} 2^2$

$$4 = 4 \checkmark$$

$n=3$: $(2\cdot 1-1)+(2\cdot 2-1)+(2\cdot 3-1) \stackrel{?}{=} 3^2$

$$9 = 9 \checkmark \quad \text{Equality holds.}$$

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2. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

a) Take base case $n=1$: $1 \stackrel{?}{=} \frac{1(1+1)}{2}$

$1 = 1$ ✓.

Now assume equality holds for $(n-1)$.

Then: $\sum_{k=1}^{n-1} k = 1+2+\ldots+(n-1) = \frac{(n-1)((n-1)+1)}{2} = \frac{(n-1)n}{2}$

Now consider case n:

$$\sum_{k=1}^{n} k = 1+2+\ldots+(n-1)+n \stackrel{?}{=} \frac{n(n+1)}{2}$$

$$= \left(\sum_{k=1}^{n-1} k\right) + n$$

$$= \frac{(n-1)n}{2} + n = \frac{n^2-n}{2} + \frac{2n}{2} = \frac{n^2+n}{2}$$

$$= \frac{n(n+1)}{2}.$$ Equality holds.

b) Gauss's method:

$\sum_{k=1}^{n} k = \frac{n(u+1)}{2}$

$$\begin{array}{rccccccccc} \sum_{k=1}^{n} k = & 1 & + & 2 & + & 3 & + & \ldots & + & \frac{n}{2} \\ & n & + & n-1 & + & n-2 & + & \ldots & + & \frac{n}{2}+1 \\ \hline & (n+1) & + & (n+1) & + & (n+1) & + & \ldots & + & (n+1) \end{array}$$

Therefore there are $\frac{n}{2}$ pairs of $(n+1)$ and

$$\sum_{k=1}^{n} k = \frac{n}{2}(n+1)$$

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c. Dr. Z's way. Need to prove to $n=3$, since $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ is a polynomial of degree 2.

Case $n=1$ $\quad 1 \stackrel{?}{=} \frac{1(1+1)}{2}$

$1 = 1$ ✓

$n=2$ $\quad 1+2 = \frac{2\cdot(2+1)}{2}$

$3 = 3$ ✓

$n=3$ $\quad 1+2+3 = \frac{3\cdot(3+1)}{2}$

$6 = 6$ ✓ Equality holds

3. $\sum_{k=1}^{n} k^2$, since the summands are of degree 2, then the sum will be of degree 3, $A(n) = an^3 + bn^2 + cn$

$A(1) = 1 \qquad A(2) = 1 + 2^2 = 5 \qquad A(3) = 1 + 2^2 + 3^2 = 14$

then

$$\left.\begin{array}{l} A(1) = a\cdot 1 + b\cdot 1 + c\cdot 1 = 1 \\ A(2) = a\cdot 8 + b\cdot 4 + c\cdot 2 = 5 \\ A(3) = a\cdot 27 + b\cdot 9 + c\cdot 3 = 14 \end{array}\right\} \begin{array}{l} \to 6a + 2b = 3 \\ \\ \to 24a + 6b = 11 \end{array} \Big] \to \begin{array}{l} -2b = -1 \\ b = \frac{1}{2} \end{array}$$

$c = \frac{1}{6} \qquad a = \frac{1}{3}$

Therefore: $\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n}{6}(2n+1)(n+1)$

a) Take base case: $n=1$

$\sum_{k=1}^{1} k^2 = 1^2 \stackrel{?}{=} \frac{1}{3} + \frac{1}{2} + \frac{1}{6}$

$1 = 1$ ✓

Now assume equality holds for $(n-1)$. Then

$$\sum_{k=1}^{n-1} k^2 = \frac{(n-1)}{6}(2(n-1)+1)(n-1+1)$$
$$\frac{(n-1)}{6}(2n-1)\cdot n$$

Now consider case n. Then

$$\sum_{k=1}^{n} k^2 = 1+4+9+\ldots(n-1)^2+n^2 \stackrel{?}{=} \frac{n}{6}(2n+1)(n+1)$$
$$= \left(\sum_{k=1}^{n-1} k^2\right) + n^2$$
$$= \frac{1}{6}(n-1)(2n-1)\cdot n + n^2$$
$$= \frac{1}{6} n [2n^2 - 3n + 1] + \frac{n}{6}(6n)$$
$$= \frac{n}{6}[2n^2 - 3n + 1 + 6n]$$
$$= \frac{n}{6}[2n^2 + 3n + 1]$$
$$= \frac{n}{6}(2n+1)(n+1).$$ Equality holds.

6) Dr Z's way. Need to show that the equality holds at 4 different values for degree 3:

$n=1$ $\quad 1 \stackrel{?}{=} \frac{1}{6}(2+1)(1+1)$

$1 = 1$ ✓

$n=2$ $\quad 1+2^2 = \frac{2}{6}(2\cdot 2+1)(2+1)$

$5 = 5$ ✓

$n=3$ $\quad 1+2^2+3^2 = \frac{3}{6}(2\cdot 3+1)(3+1)$

$14 = 14$ ✓

$n=4$ $\quad 1+2^2+3^2+4^2 = \frac{4}{6}(2\cdot 4+1)(4+1)$

$30 = 30$ ✓

4 $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

The sum is a polynomial of degree 4. Therefore, need to make sure that the equality agrees at 5 points:

$n=1$: $1^3 \stackrel{?}{=} \left(\frac{1(1+1)}{2}\right)^2$

$1 = 1$ ✓

$n=2$ $1^3+2^3 \stackrel{?}{=} \left(\frac{2(2+1)}{2}\right)^2$

$9 = 9$ ✓

$n=3$ $1^3+2^3+3^3 \stackrel{?}{=} \left(\frac{3(3+1)}{2}\right)^2$

$36 = 36$ ✓

$n=4$ $1^3+2^3+3^3+4^3 \stackrel{?}{=} \left(\frac{4(4+1)}{2}\right)^2$

$100 = 100$ ✓

$n=5$ $1^3+2^3+3^3+4^3+5^3 \stackrel{?}{=} \left(\frac{5(5+1)}{2}\right)^2$

$225 = 225$ ✓

Therefore, equality holds.