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HW 3 <u>OK to Post</u>

1) Prove

$$P(n) = \sum_{k=1}^{n} 2k-1 = n^2$$

a) Base case $n=1$

$P(1) = 2(1) - 1 = 1^2$

$1 = 1$

Assuming it is true for $(n-1)$ then it is true for n

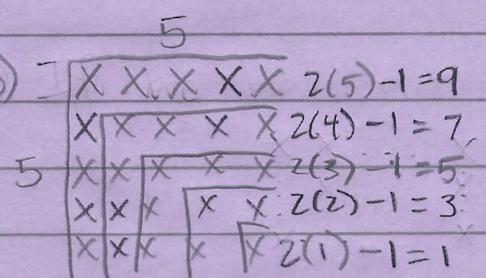
$P(n) = 1+3+\dots+(2n-1) = n^2$

Show $P(n+1) = 1+3+\dots+(2n-1)+(2n+1) = (n+1)^2$

For all n, if $1+3+5\dots+(2n-1) = n^2$ then,

$1+3+\dots+(2n-1)+(2n+1) = (n+1)^2$

For all n, $1+3+\dots+(2n-1) = n^2$

b) 

$\rightarrow \sum_{k=1}^{n} 2k-1 = n^2$

c) $P(n) = n^2$ is a polynomial of degree 2 therefore they must coincid in $(2+1)=3$ places to be true

$P(1) = 1 = 1^2 = 1$

$P(2) = 1+3 = 2^2 = 4$

$P(3) = 1+3+5 = 3^2 = 9$

From this we know $\sum_{k=1}^{n} 2k-1 = n^2$ for all n

2) Prove $P(n) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

a) Base case $n=1$

$$1 = \frac{1(1+1)}{2} = 1$$

Assuming it is true for $n-1$ then it is true for n

$$P(n-1) = 1+2+\ldots+(n-1) = \frac{(n-1)(n\cancel{+1}\cancel{-1})}{2} = \frac{n(n-1)}{2}$$

$$P(n) = 1+2+\ldots+(n-1)+n = \frac{(n-1)n}{2} + n = n\left(\frac{n-1}{2}+1\right)$$

$$= n\left(\frac{n-1+2}{2}\right) = \frac{n(n+1)}{2}$$

b) $S(n) = 1+2+\ldots n$

Add $S(n)$ to both sides

$$S(n) = 1+2+\ldots+n$$

$$S(n) = n+n-1+\ldots+3+2+1 \quad +$$

$$2S(n) = (n+1)+(n+1)+(n+1)\ldots+(n+1) = n(n+1)$$

Solve for $S(n)$

$$S(n) = \frac{n(n+1)}{2}$$

c) If $p(n)$ is a polynomial of degree K then

$$S(n) = p(1)+p(2)+\ldots+p(n) \Rightarrow$$

$$S(n) = \sum_{i=1}^{n} P(i)$$

is automatically a polynomial of degree $K+1$

If 2 polynomial of degree K coincide in $K+1$ different places, they are always the same.

$P(1) = 1 = \frac{1(1+1)}{2} = 1$

$P(2) = 1+2 = \frac{2(2+1)}{2} = 3$ From this we know: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$P(3) = 1+2+3 = \frac{3(3+1)}{2} = 6$ for all n

3) Derive explicit formula: $\sum_{k=1}^{n} k^2 = p(n)$

We are going to start with the binomial expansion of $(k-1)^3$

$$(k-1)^3 = k^3 - 3k^2 + 3k - 1$$

Rearrang the terms

$$k^3 - (k-1)^3 = 3k^2 - 3k + 1$$

We are going to then do the infinite sums of both sides from $k=1$ to n

$$\sum_{k=1}^{n} (k^3 - (k-1)^3) = 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

The LHS is a telescoping series resulting in n^3, and w know by Gauss' formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} 1 = n$

$$n^3 = 3\sum_{k=1}^{n} k^2 - \cancel{3\frac{n(n+1)}{2}} + \cancel{n}$$

$$+\cancel{3\frac{n(n+1)}{2}} - \cancel{n}$$

$$\left(\cancel{3}\sum_{k=1}^{n} k^2 = n^3 + 3\frac{n(n+1)}{2} - n\right) \frac{1}{3}$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{n(n+1)}{2} - \frac{1}{3}n$$

$$\Rightarrow \sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{n(n+1)}{2} - \frac{1}{3}n$$

4) Prove that $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 = P(n)$

$P(n)$ is a function of degree 4 which means we need $4+1$ inputs to show that it is true for all $P(n)$

$P(1) = 1 = \left(\frac{1(1+1)}{2}\right)^2 = 1$

$P(2) = 1 + 8 = \left(\frac{2(2+1)}{2}\right)^2 = 9$

$P(3) = 1 + 8 + 27 = \left(\frac{3(3+1)}{2}\right)^2 = 36$

$P(4) = 1 + 8 + 27 + 64 = \left(\frac{4(4+1)}{2}\right)^2 = 100$

$P(5) = 1 + 8 + 27 + 64 + 125 = \left(\frac{5(5+1)}{2}\right)^2 = 225$