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It is OK to post the homework in your web-site

1. (a) $1-\frac{5}{6}\times\frac{5}{6}=\frac{11}{36}$

(b) $1-(\frac{5}{6})^{10}$

(c) $1-(\frac{5}{6})^{n}$

2. Prove by induction.

Base case: Choosing 1 element out of a set of n elements, it is obvious that there are n ways.

When $k=1$, $\frac{n!}{1\cdot(n-1)!}=n$, which is true.

Assume that choosing k elements out of a set of n elements, there are $\frac{n!}{k!(n-k)!}$ elements.

Then
$$\binom{n}{k}+\binom{n}{k-1}=\frac{n!}{k!(n-k)!}+\frac{n!}{(k-1)!(n-k-1)!}$$
$$=\frac{n!(n-k+1)}{k!(n+1-k)!}+\frac{kn!}{k!(n+1-k)!}$$
$$=\frac{(n+1-k+k)n!}{k!(n+1-k)!}=\frac{(n+1)!}{k!(n+1-k)}=\binom{n+1}{k}$$

3. We have known that "n choose k" represent as $\binom{n}{k}=\frac{n!}{k!(n-k)!}$

When p is the probability of each choice we want, k is the number of choices we want, n is the total number of choice. Then we have the probability is $p^k(1-p)^{(n-k)}$, But we need to include the total number of outcomes, so the probability is $\binom{n}{k}p^k(1-p)^{n-k}$.

4. Suppose $\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}dx=k$, $\int_{-\infty}^{\infty}e^{-\frac{y^2}{2}}dy=k$,
$$k^2=\left(\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}dx\right)\left(\int_{-\infty}^{\infty}e^{-\frac{y^2}{2}}dy\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{x^2+y^2}{2}}dxdy.$$

Suppose $x^2+y^2=r^2$, $r\in[0,\infty)$, $\theta\in[0,2\pi)$,

then $k^2=\int_0^{\infty}\int_0^{2\pi} re^{-\frac{r^2}{2}}\,d\theta\,dr.=\int_0^{\infty} re^{-\frac{r^2}{2}}\left(\int_0^{2\pi}d\theta\right)dr$

$=\int_0^{\infty} re^{-\frac{r^2}{2}}(2\pi)\,dr=2\pi\int_0^{\infty} re^{-\frac{r^2}{2}}\,dr.$

$\int re^{-\frac{r^2}{2}}=-e^{-\frac{r^2}{2}}+C$, $\int_0^{\infty} re^{-\frac{r^2}{2}}\,dr=-e^{\frac{-r^2}{2}}\Big|_0^{\infty}=1$

Then $k^2=2\pi$, $k=\sqrt{2\pi}$. Then $\int_{-\infty}^{\infty}\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}=\frac{k}{\sqrt{2\pi}}=1$,

which means $\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ is a probability density function.