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Homework for Lecture 19 - OK to post

① a.) $n=2$ since we roll a die twice
$p=\frac{1}{6} \longrightarrow$ probability of success (getting a 1)
$1-p=\frac{5}{6}$

P(at least one 1) = 1 − P(none are a 1)

We use the formula $\binom{n}{k}p^k(1-p)^{n-k}$, where k is the number of successes, so in this case, $k=0$. Thus

$$P(\text{at least one } 1) = 1-\binom{2}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^2 = \frac{11}{36} \approx 30.6\%$$

b.) $n=10$ since we roll a die ten times
$p=\frac{1}{6} \rightarrow$ probability of success (getting a 1)
$1-p=\frac{5}{6}$

P(at least one 1) = 1 − P(none are 1)

Using the same formula where $k=0$, we see that

$$P(\text{at least one } 1) = 1-\binom{10}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{10} \approx 83.8\%$$

c.) We use the same formula as above, but since n is unknown, we leave it in terms of n. Thus

$$P(\text{at least one } 1) = 1-P(\text{none are } 1) = 1-\binom{n}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^n = 1-\left(\frac{5}{6}\right)^n$$

② We use the Taylor expansion formula:

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \dots + f^{(k)}(0)\frac{x^k}{k!}$$

Let $f(x) = (1+x)^n$. Then

$f(0) = 1$
$f'(x) = n(1-x)^{n-1}$
$f'(0) = n$
$f''(x) = n(n-1)(1+x)^{n-2}$
$f''(0) = n(n-1)$
$f^{(k)}(x) = n(n-1)(n-2)\cdots(n-(k-1))(1+x)^{n-k}$
$f^{(k)}(0) = n(n-1)(n-2)\cdots(n-(k-1))$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)(n-(k-1))}{k!}x^k$$

So the binomial coefficient formula consists of the coefficients of the terms of the expansion above, so

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k!}$$

This can be rewritten as

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k!} \cdot \frac{(n-k)!}{(n-k)!}$$

So

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

(3) Since we are flipping the coin n times and we want k heads (which are our successes), there are many ways that this can happen. We need to count the number of ways of ordering the successes and failures, and for that, we use the binomial coefficient since the order of successes and failures does not matter. Thus we can say

$$\binom{n}{k}, \text{ which is } n \text{ trials, choose } k \text{ successes.}$$

Then we see what the probability of success is. Let p be the probability of getting a head (success). Then the probability of a tail (failure) is $(1-p)$. Since we want k successes, and there are n total trials, there must be $n-k$ failures. Thus we obtain

$$\binom{n}{k} p^k (1-p)^{n-k}$$

as the formula to finding probability of exactly k heads when flipped n times.

(4) Let $a = \int_{-\infty}^{\infty} e^{-x^2/2} \, dx$

Then $a^2 = \left(\int_{-\infty}^{\infty} e^{-x^2/2} \, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2} \, dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \, dx\,dy$

Then in polar coordinates, we have

$$\begin{aligned} a^2 &= \int_0^{\infty}\int_0^{2\pi} re^{-r^2/2} \, d\theta\, dr \\ &= \int_0^{\infty} re^{-r^2/2}\left(\int_0^{2\pi} d\theta\right) dr \\ &= \int_0^{\infty} re^{-r^2/2}(2\pi) \, dr \end{aligned}$$

$$a^2 = 2\pi \int_0^\infty r e^{-r^2/2}\, dr$$

We use u-substitution where $u = \frac{-r^2}{2}$. Then $du = -r$, so we have

$$-\int e^u du = -e^{-r^2/2} + C \quad \text{since } \int e^u = e^u$$

Therefore

$$\left[-e^{-r^2/2}\right]_0^\infty = -\frac{1}{e^\infty} - (-e^0) = 0 - (-1) = 1$$

So $a^2 = 2\pi(1)$, so $a^2 = 2\pi$ and $a = \sqrt{2\pi}$.

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