Sarah Magno
Dr. Z, History of Math
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Homework for Lecture 18 - OK to post

① The five Platonic solids are:

Tetrahedron: $V=4, E=6, F=4$

Hexahedron: $V=8, E=12, F=6$

Octahedron: $V=6, E=12, F=8$

Dodecahedron: $V=20, E=30, F=12$

Icosahedron: $V=12, E=30, F=20$

② We use induction on the number of edges.

First, suppose that the graph is not a tree, so there is at least one cycle. Pick any edge along the cycle and delete it. This makes a smaller graph where the number of edges and faces both decrease by 1. Let v', e', and f' represent the number of vertices, edges, and faces respectively in the new smaller graph. Then

$$v'=v \; ; \; e'=e-1 \; ; \text{ and } f'=f-1$$

Thus $v-e+f = v'-e'+f'$, so by the inductive hypothesis, it is 1.

The initial base case is a tree. A tree is a connected graph with no cycles. In a tree, the number of regions, f, is 0 since we are not counting the "infinite ocean."

In the tree, pick any leaf, which is a vertex of degree 1, and delete it. In this smaller graph, the number of vertices and edges both decrease by 1, so we have

$$v' = v-1 \; ; \; e' = e-1 \; ; \; f' = 0$$

We see that $v-e+f = v'-e'+f'$, and this is 1 by the inductive hypothesis.

The ultimate base case is the trivial tree with one vertex, so there are no edges and no regions (since we aren't counting the "infinite ocean.")

$$v = 1 \; ; \; e = 0 \; ; \; f = 0$$

So obviously $v-e+f = 1-0+0 = 1$.

(3) (i.) Since every vertex has a edges coming out of it, and there are v vertices, then we see that there are aV edges. But since every edge belongs to 2 vertices, then $2E = aV$, so $E = \frac{aV}{2}$ and thus $V = \frac{2E}{a}$

Since every face has b edges around it, and there are F faces, then we see that there are bF edges. But since every edge belongs to 2 faces, then $2E = bF$, so $E = \frac{bF}{2}$ and thus $F = \frac{2E}{b}$

(ii.) Using the formula $V-E+F=2$, we plug in to obtain

$$\frac{2E}{a} - E + \frac{2E}{b} = 2$$

Factoring E out, we get

$$E\left(\frac{2}{a} - 1 + \frac{2}{b}\right) = 2$$

So $E = \dfrac{2}{\frac{2}{a} - 1 + \frac{2}{b}}$

(iii) The only values of $3 \le a, b \le 5$ where a, b are positive integers that work are:

$a = 3, b = 3$: Tetrahedron with $V = 4$, $E = 6$, $F = 4$

$a = 3, b = 4$: Hexahedron with $V = 8$, $E = 12$, $F = 6$

$a = 3, b = 5$: Dodecahedron with $V = 20$, $E = 30$, $F = 12$

$a = 4, b = 3$: Octahedron with $V = 6$, $E = 12$, $F = 8$

$a = 5, b = 3$: Icosahedron with $V = 12$, $E = 30$, $F = 20$

4) A soccer ball is not perfect because all of its faces are not the same polygon – it is made up of a mixture of hexagons and pentagons

$V = 60$
$E = 90$
$F = 32$ (12 pentagons + 20 hexagons)

$V - E + F = 60 - 90 + 32 = 2$ ✓