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Homework for Lecture 16 - do not post

(1a.) There are 12 members of A_4:

$$\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix} \begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix} \begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix} \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix} \begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix} \begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix} \begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix} \begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix} \begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix} \begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$$

A_4 is a subgroup of S_4 because it contains the identity element, and it is closed under multiplication, since by a lemma, the product of an even permutation and an even permutation is an even permutation. Also, the inverse of an even permutation is an even permutation, since $a * a^{-1} = I$, and a and I are even, so by the lemma, a^{-1} is even.

(1b.) We must show that H is closed under multiplication and each member has an inverse that is in the set.

(i) The first member is the Identity element, so when it is multiplied by the other two members, it returns those two members which are in the set. Also,

$$\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}$$ which is in the set

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(ii.) The inverse of the Identity element is the Identity element, which is in the set.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}^{-1} \Rightarrow \begin{pmatrix} 2 & 3 & 1 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \text{ which is in the set}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}^{-1} \Rightarrow \begin{pmatrix} 3 & 1 & 2 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \text{ which is in the set}$$

1c. The first coset is H itself, since eH is a coset.

Next we kick out the members of H to form the cosets.

One coset is $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}H$, which consists of

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

$$\left\{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}\right\}$$

Kicking those members out, we form another coset $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}H$, which consists of

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

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$$\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix}=\begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}$$

So the coset is $\left\{\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix},\begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix},\begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}\right\}$

Kicking those members out, we form another coset $\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}H$, which consists of

$$\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}=\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}=\begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix}=\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}$$

So the coset is $\left\{\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix},\begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix},\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}\right\}$

When we kick out those members, we see that nothing is left, so there are only 4 left cosets.

They all have the same number of elements since the relation that takes the set H to its cosets is a one-to-one correspondence with inverses, so $|aH|=|H|=|bH|$ for all cosets a, b, etc.

Two cosets can't have a common element. For the sake of contradiction, suppose that $a*H$ has a common element with $b*H$.

Multiply on the left by a^{-1} to obtain

$h_1 = a^{-1} * b * h_2$

Now multiply from the right by h_2^{-1}

$h_1 h_2^{-1} = a^{-1} * b$

Therefore $a^{-1} * b$ belongs to H and therefore a belongs to the coset $b * H$, so $a * h = b * H$. So if they have one element in common, then they are identical. Otherwise, they have no elements in common.

(1d) From the previous problem, we saw that

$$A_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} H \cup \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} H \cup \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} H \cup \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} H$$

This is a left coset decomposition of the larger group A_4 into cosets of H. In the previous problem, we also showed that each of them have the same number of elements with nothing in common. Therefore, $\frac{|A_4|}{|H|}$ must be an integer

(2) If G is a group with $|G|$ elements and H is a subgroup of G with $|H|$ elements, then $|G|/|H|$ is always an integer.

Let G have m elements and H have n elements. Then $H = \{h_1, \ldots, h_n\}$ with the assumption that $h_1, \ldots, h_n$ are all distinct. If we let $G = H$, then since $m/n = 1$, this demonstrates that case. If $G \neq H$, then there exists an element $g_1 \in G \setminus H$. Let the left coset $g_1 H = \{g_1 h_1, \ldots, g_1 h_n\}$. The members $g_1 h_1, \ldots, g_1 h_n$ are all distinct. For the sake of contradiction, suppose they are not. Then for some $1 \leq i < j \leq n$, we have $g_1 h_i = g_1 h_j$. Multiply both sides on the left by g_1^{-1} to obtain $h_i = h_j$. But we said earlier that $h_1, \ldots, h_n$ are distinct, so contradiction. We also know that $g_1 H$ and H have nothing in common. Again, for the sake of contradiction, suppose that they did. Then for some h_i, h_j in H, we have

$$g_1 h_i = h_j$$

Multiply both sides on the right by h_i^{-1} to get

$$g_1 = h_j h_i^{-1}$$

But since H is a group, this shows that $g_1 \in H$, but this is a contradiction. If we let $G = H \cup g_1 H$, then this satisfies the condition. If not, let $g_2 \in G$ where $g_2 \notin H$ and $g_2 \notin g_1 H$. Using the same strategy as above, all of $g_2 H$'s elements are distinct and have nothing in common with H and $g_1 H$. Keep doing this until we obtain

$$G = H \cup g_1 H \cup g_2 H \cup \dots \cup g_{r-1} H$$

Each has n elements with nothing in common, so $m = nr$, so $r = \frac{m}{n}$ is an integer.

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