MATH 437 · HOMEWORK 16 NINA CHALGERI

① (a) ALL PERMUTATIONS

$\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&2&4&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix},$

$\begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix},$

$\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&4&1&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}, \Rightarrow$

$\begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&1&4&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix},$

$\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&4&2&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix},$

$\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&2&3&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&3&1&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$

Every other permutation is even

12 EVEN PERMUTATIONS ←

EVEN PERMUTATIONS

(1 2 3 4)
(1 3 4 2)
(1 4 2 3)
(2 1 4 3)
(2 3 1 4)
(2 4 3 1)
(3 1 2 4)
(3 2 4 1)
(3 4 1 2)
(4 1 3 2)
(4 2 1 3)
(4 3 2 1)

· A_4 has 12 elements.

* A_4 is a subgroup of the symmetric group on $\{1, 2, 3, 4\}$.

PROOF.

· We want to check that A_4 satisfies all axioms of what defines a subgroup.

(i) A_4's identity element is ~~{1,2,3,4}~~ (1 2 3 4)

(ii) If we multiply any two elements in A_4, the product of the two elements will also exist in A_4 because multiplying two matrices, ~~with~~ both with even inversons will give a matrix with an even number of inversions.

(iii) The inverse of any element in A_4 is also in A_4.

$a = \begin{pmatrix}a&b&c&d\\a&c&d&b\end{pmatrix} \quad a^{-1} \Rightarrow \begin{pmatrix}a&c&d&b\\a&b&c&d\end{pmatrix} \Rightarrow \begin{pmatrix}a&b&c&d\\a&d&b&c\end{pmatrix} = a^{-1} \Rightarrow a^{-1}$ also has an even number

(iv) the multiplication associativity applies

$\begin{pmatrix}a&b&c&d\\a&c&d&b\end{pmatrix}\left[\begin{pmatrix}a&b&c&d\\a&b&c&d\end{pmatrix} \cdot \begin{pmatrix}a&b&c&d\\a&d&b&c\end{pmatrix}\right] = \left[\begin{pmatrix}a&b&c&d\\a&c&d&b\end{pmatrix} \cdot \begin{pmatrix}a&b&c&d\\a&b&c&d\end{pmatrix}\right] \cdot \begin{pmatrix}a&b&c&d\\a&d&b&c\end{pmatrix}$ ∎

(b) $H = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \right\}$ SHOW $H \leq A_4$

(i) H contains the identity permutation: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ ✓

Satisfies inverses

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}^{-1} \Rightarrow \begin{pmatrix} 2 & 3 & 1 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$ ✓ $\in H$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}^{-1} \Rightarrow \begin{pmatrix} 3 & 1 & 2 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ ✓ $\in H$

(iii) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \in H$

satisfies multiplicity

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \in H$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \in H$

(iv) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \cdot \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \right] = \left[\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \right] 2$

$\times \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

(c)

(c) $|A_4| = 12$ and $|H| = 3$ $\Rightarrow$ 4 left cosets of H.

~~A₄ without~~ $\{A_4\} - \{H\}$ $\Rightarrow$ left with nine matrices

$\Rightarrow \begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&4&2&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\2&4&3&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&2&4&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix} \begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}$

$\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$ $\Rightarrow$ multiplying every element by H ~~gives us~~ and removing them from the set : $\{A_4 - H\}$ = call this set B

~~$\{A_4 - H\}$ = B~~ NOT SURE WHAT TO DO HERE...

(d) NOT SURE... :(

~~② Lagrange's Theorem:~~
~~If H is a subgroup of a group G, then |G| = [G:H] |H|.~~
~~PROOF.~~
~~· If H is a subgroup of G and K is a subgroup of H, then~~
~~[G:K] = [G:H][H:K]~~
~~· Let S be a set of coset representatives for H in G, so~~

② * Lagrange's Theorem: Let H be a subgroup of a finite group G. Then the order of H divides the order of G.

PROOF.

- Let $aH = \{ah \mid h \in H\}$ be the left coset of H containing a.
  - * Lemma: Every coset of H has the same number of elements as H.
    - PROOF. Want to show a function mapping $H \to aH$ is a bijection
      - Let $f: H \to aH$ such that $f(h) = ah$ → (injection)
      - if $f(h_1) = f(h_2)$ then $ah_1 = ah_2$ so therefore $h_1 = h_2$
      - This is a surjection because for every element in H, call h_k, then $f(h_k) = ah_k$. Since every element is unique, so is every element in aH.
      - Therefore, $f: H \to aH$ is a bijection and thus $|aH| = |H|$. ■
- Let G have n elements and H have m elements.
- Let $H, a_1H, a_2H, \dots, a_{k-1}H$ be the k distinct cosets of H
- Each coset has the same cardinality as H and each element of G appears in exactly one coset so:

$$n = mk$$

- Hence, m divides n, and therefore, the order of H divides the order of G. ■