History of Math | Homework 16 | Karan Amin

① All permutations [1,2,3,4]

[1,2,3,4] *
[2,1,3,4]
[3,1,2,4] *
[1,3,2,4]
[2,3,1,4] *
[3,2,1,4]
[3,2,4,1] *
[2,3,4,1]
[4,3,2,1] *

[3,4,2,1]
[2,4,3,1] *
[4,2,3,1]
[4,1,3,2] *
[1,4,3,2]
[3,4,1,2] *
[4,3,1,2]
[1,3,4,2] *

[3,1,4,2]
[2,1,4,3] *
[1,2,4,3]
[4,2,1,3] *
[2,4,1,3]
[1,4,2,3] *
[4,1,2,3]

All permutations with * are even permutations. Let all those permutations be in A_4.

Show that A_4 is a subgroup of the symmetric group {1,2,3,4}

- Since $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ is in A_4, A_4 contain the identity.

Next show closure

- Every permutation in A_4 has an even number of inversions, 0 or 2. Then the product of any two permutation will also have an even number of inversions, hence will also be in A_4.

Therefore A_4 is a subgroup.

b) $H = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \right\}$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \in H$, hence H contains the identity ~~ete~~ permutation. Then to show closure, let us simply ~~con~~ calculate all products

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \in H$

$(1\ 2\ 3\ 4) \cdot (2\ 3\ 1\ 4) = (2\ 3\ 1\ 4) \in H$

$(1\ 2\ 3\ 4) \cdot (3\ 1\ 2\ 4) = (3\ 1\ 2\ 4) \in H$

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hence H is a subgroup

② Lagrange's Theorem: Let H be a subgroup of G, where G is a finite group. Then the order or number of elements of H divides the order of G. Or in other words $|G| / |H|$ is an integer.

We will use the lemma that every right coset of H in G has the same order as H.

Let $aH = \{ah \mid h \in H\}$ be the left cosets of H. Let G have n elements and its subgroup H have m elements. Then $a_1H, a_2H, \ldots, a_nH$ be the unique distinct cosets of H in G.

We know form the lemma that each coset has the same cardinality as H and each element of G appears in one of the cosets. Then $n = mp$ for some integer p, hence the number of elements in H divides the number of elements in G.