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Homework for Lecture 15 - do not post

① The 6 permutations of $\{1,2,3\}$ are

$\{1,2,3\} \to 0$ inversions

$\{1,3,2\} \to 1$ inversion

$\{2,1,3\} \to 1$ inversion

$\{2,3,1\} \to 2$ inversions

$\{3,1,2\} \to 2$ inversions

$\{3,2,1\} \to 3$ inversions

② 1523746

1253746

1235746

1235476

1234576

1234567

Therefore, it took 5 inversions to arrive at the final state.

(3) Let 16 = the blank spot in the puzzle. We will use a lemma: The number of inversions that is changed in one legal move is always odd. Next we set up the invariant S, where

$$S = \text{number of inversions} + \text{taxicab distance from } 16$$

where taxicab distance = row number + column number − 2 of the blank spot (16). This calculates the distance to the top left hand corner.

By using the lemma, in any legal move (horizontal or vertical), S does not change parity, since the number of inversions changes by an odd number, and the taxicab distance changes by ± 1 since either the row or column number changed by 1. Therefore, if it starts even, it stays even, and if it starts odd, it stays odd.

Initial State: $S = 0 + (4+4-2) = 6$

Final State: $S = \underbrace{1}_{\text{since the 14 and 15 changed spots but the 16 stayed where it was}} + (4+4-2) = 7$

Since 6 is even and 7 is odd, it is impossible by sliding to get to the final state.

(4) A group G satisfies 4 properties under an operation called multiplication, denoted by $*$ such that

(i.) For any members g, g' of G, $g * g'$ is also in G

(ii.) There exists a "special number," called e (the identity member) such that $g * e = g$ and $e * g = g$ for every member of G.

(iii.) (Associativity) For any three members of G, g_1, g_2, g_3,
$(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$

(iv.) Every member of G has an inverse, denoted by g^{-1}, such that

$$g * g^{-1} = e \text{ and } g^{-1} * g = e$$

(5) We show that the set of all 2x2 matrices with integer entries and determinant 1 is a group by showing that the four properties hold. Let G be the name of the group and let $A, B \in G$.

(i.) Since $\det(AB) = \det(A)\det(B) = 1 \cdot 1 = 1$, AB is in G as well, showing that it is closed under multiplication.

(ii.) Associativity → known from Linear Algebra

(iii.) The identity element is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$. This matrix is in G because it is 2x2, it has integer entries, and its determinant is 1. It satisfies $A * I_2 = A$ and $I_2 * A = A$ for every $A \in G$.

(iv.) The inverse for any matrix in G can be found by the following formula:

$$\frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

This matrix is in G for all members of G because since $\det(A) = 1$, it can be reduced to

$$\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

and the determinant of this matrix is $a_{22}a_{11} - a_{21}a_{12}$, which is equivalent to $a_{11}a_{22} - a_{12}a_{21}$ which is 1. Also, it satisfies $A * A^{-1} = I_2$ and $A^{-1} * A = I_2$ for every $A \in G$.

In this case, G is an infinite group.

⑥ We show that the set of all 2x2 matrices with entries in $\{0,1,2\}$ with non-zero determinant and matrix multiplication mod 3 is a group by showing that the four properties hold. Let G be the name of the group and let $A, B \in G$.

(i.) Since $\det(A) \neq 0$ and $\det(B) \neq 0$, then $\det(AB) = \det(A)\det(B) \neq 0$. So AB has nonzero determinant, and since the multiplication is done mod 3, all its entries will be either 0, 1, 2, so AB will be in G for all A, B in G.

(ii.) Associativity → known from Linear Algebra

(iii.) The identity element is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$. This matrix is in G because all its entries are either 0, 1, or 2 and its determinant is 1, which is nonzero. Also $A * I_2 = A$ and $I_2 * A = A$ for all A in G.

(iv.) Since all elements of G have non-zero determinants, this means that they are invertible, and their inverses can be found by the following formula

$$\frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Now we need to show that the inverses are all in the group, but it is not clear what the next step is, as fractions may appear, which cause the matrices to not have 0, 1, or 2 as entries.

It is a finite group with 48 elements. Below are the only possible 2x2 matrices with entries $\{0,1,2\}$ and non-zero determinant:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$$

7 As per Dr. Z's comment at the end of the Nov. 3 class, we are postponing this question to the next homework.