Sarah Magno
Dr. Z, History of Math
10/24/21

Homework for Lecture 13 - OK to post

① We use the Taylor series expansion of arctan x, which is

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{1}{9} - \ldots$$

We plug in 1 for x, obtaining

$$\arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

Since $\arctan(1) = \frac{\pi}{4}$, we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

Multiplying both sides by 4, we see that

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right)$$

which proves Leibniz' formula.

①' The first 10 non-zero terms of the formula are

$$4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19}\right)$$

$$= 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \frac{4}{15} + \frac{4}{17} - \frac{4}{19}$$

$= 3.041839619$ as an approximation for π

We use the fact $|\text{error}| < |a_{n+1}|$, where a_{n+1} is the first neglected term.

Since $a_{10} = \frac{4}{21}$ is the first neglected term, we can estimate that the error will be less than $\frac{4}{21}$, or $0.\overline{190476}$.

② We use the formula

$$\arctan(x) + \arctan(y) = \arctan \frac{x+y}{1-xy}$$

where $x = \frac{1}{2}$ and $y = \frac{1}{3}$ to obtain

$$\arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right) = \arctan \frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}$$

$$= \arctan \frac{\frac{5}{6}}{\frac{5}{6}}$$

$$= \arctan(1)$$

$$= \frac{\pi}{4}$$

This proves Euler's formula, which is $\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$.

②' We rewrite Euler's formula as

$$\pi = 4\arctan\tfrac{1}{2} + 4\arctan\tfrac{1}{3}$$

Now we use the Taylor expansion of $\arctan x$ to expand the right hand side

$$4\left(\tfrac{1}{2} - \tfrac{1}{24} + \tfrac{1}{160} - \cdots\right) + 4\left(\tfrac{1}{3} - \tfrac{1}{81} + \tfrac{1}{1215} - \cdots\right)$$

$= 3.1415576132$, or 3.14 as a two digit approximation for π

(3) We rewrite Machin's formula as

$$\pi = 16\arctan\left(\tfrac{1}{5}\right) - 4\arctan\left(\tfrac{1}{239}\right)$$

Now we use the Taylor expansion of $\arctan x$ to expand the right hand side

$$16\left(\tfrac{1}{5} - \tfrac{1}{375} + \tfrac{1}{15625} - \ldots\right) - 4\left(\tfrac{1}{239} - (2.44165918 \cdot 10^{-8}) + (2.56472314 \cdot 10^{-13}) - \ldots\right)$$

$= 3.141621029$, or 3.141 as a three digit approximation for π

(4) The first four terms of Jesus Guillera's formula are

$$13 - \left(\tfrac{1}{2}\right)^5 \frac{20 \cdot 50 + 13}{2^{10}} + \left(\tfrac{1 \cdot 3}{2 \cdot 4}\right)^5 \frac{40 \cdot 91 + 13}{2^{20}} - \left(\tfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^5 \frac{60 \cdot 132 + 13}{2^{30}}$$

which gives $12.9691150\ldots$ as an approximation for $\frac{128}{\pi^2}$

Call x the approximation we found above. We can use this to get an approximation for π by solving

$$x \approx \frac{128}{\pi^2}$$

$$\pi^2 \approx \frac{128}{x}$$

$$\pi \approx \sqrt{\frac{128}{x}}$$

so $\pi \approx 3.14159265359 21\ldots$ Since π is actually equal to $3.14159265358 9\ldots$, the first 10 digits agree with the true value of π.