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Homework 13

(1) Using the fact that $\tan\left(\frac{\pi}{4}\right) = 1$ and hence $\arctan(1) = \frac{\pi}{4}$, prove Leibnitz's formula

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \cdots$$

$$\arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$$

$$\arctan(1) = \frac{\pi}{4}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots\right).$$

(2) Using the arctan addition formula, prove Euler's formula $\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$.

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right)$$

$$\arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right) = \arctan\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}\right)$$

$$\arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right) = \arctan\left(\frac{\frac{3}{6}+\frac{2}{6}}{1-\left(\frac{1}{6}\right)}\right)$$

$$\arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right) = \arctan\left(\frac{5/6}{5/6}\right)$$

$$\arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right) = \arctan(1)$$

$$\arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right) = \pi/4.$$

(3) Use Machin's formula $\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$ to approximate π to 3 digits after the decimal point.

$$\frac{\pi}{4} = 0.7853981\ldots$$

$$\pi = 4(0.785)$$

$$\boxed{\pi \approx 3.141}$$

(4) Use the first four terms of Jesus Guillera's formula to compute $128/\pi^2$. Use this to find an approximation to π.

$$\sum_{n=0}^{\infty} (-1)^n a_n \left(820n^2 + 180n\right) + 13 \frac{1}{2^{10n}} = \frac{128}{\pi^2}$$

$$= 0.40625 - 0.0309143 + \frac{3.4042}{10^6} - \frac{2.96914}{10^{11}}$$

$$= 0.37533910209030 86$$

$$\frac{128}{\pi^2} = 0.3753341020903086$$

$$\frac{128}{0.3753341020903086} = \pi^2$$

$$\pi = \sqrt{\frac{128}{0.3753341020903086}}$$