History of Math | Homework 13 | Karan Amin | You can post this

1) $\tan(\frac{\pi}{4}) = 1$ and $\arctan(1) = \frac{\pi}{4}$

We know the taylor series of $\arctan(x)$ is

$$\arctan(x) = f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{if} \quad -1 \leq x \leq 1$$

Therefore $\arctan(1) = \frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{(1)^{2n+1}}{2n+1}$

Hence $\frac{\pi}{4} = +1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$

1') To find the error we know $S = S_n + R_n$ where S_n is the sum of the first n terms and R_n is from $n+1$ to the rest. Therefor our estimated error is $|S - S_n| = |R_n|$

$S_{10} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19}\right)$

$= 3.0418\ldots$

$|S - S_{10}| = |\pi - S_{10}| = 0.09975303466$

2) Show $\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$

We know $\tan(\arctan\frac{1}{2}) = \frac{1}{2}$ and $\tan(\arctan(\frac{1}{3})) = \frac{1}{3}$

and $\tan(a+b) = \dfrac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$

therefore

$$\tan(\arctan(1/2) + \arctan(1/3)) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}} = \frac{\frac{5}{6}}{1 - \frac{1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}}$$

$= 1$

$\rightarrow \arctan(\tan(\arctan 1/2 + \arctan 1/3)) = \arctan(1) = \frac{\pi}{4}$

$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$ $\blacksquare$

2') $$\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/2)^{2n+1}}{2n+1} + \sum_{n=0}^{\infty} (-1)^n \frac{(1/3)^{2n+1}}{2n+1}$$

Let $n = 10$

$$\pi \approx 4\left(\sum_{n=0}^{10} (-1)^n \frac{(1/2)^{2n+1}}{2n+1} + \sum_{n=0}^{\infty} (-1)^n \frac{(1/3)^{2n+1}}{2n+1}\right) = 3.14159$$

$\boxed{\pi \approx 3.14}$

3) Machin's formula

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$

Let $n = 10$, then similar to before

$$\pi \approx 4\left(4\cdot\sum_{n=0}^{10}(-1)^n\frac{(1/5)^{2n+1}}{2n+1} - \sum_{n=0}^{10}(-1)^n\frac{(1/239)^{2n+1}}{2n+1}\right)$$

≈ 3.14159

$$\boxed{\pi \approx 3.142}$$

4) $$13 - \left(\frac{1}{5}\right)^5\frac{20(50)+13}{2^{10}} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^5\frac{40-91+13}{2^{20}} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^5\frac{60-132+13}{2^{30}}$$

$$= 12.99970925 \approx \frac{128}{\pi^2}$$

Therefore $\pi^2 \approx \dfrac{128}{12.99970925}$

$$\pi \approx \sqrt{\frac{128}{12.99970925}} = \underline{3}.\underline{1}37893252$$

2 digits agree with pi.