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Homework 13 10/23/2021

1. Since $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ and $\arctan(1) = \frac{\pi}{4}$:

$$\pi = 4\arctan(1)$$
$$= 4 \cdot \sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{2n+1}$$
$$= 4 \cdot \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots\right)$$

1' Then approximation to 10 nonzero terms:

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19}\right)$$
$$\approx 3.0418396\dots$$

The error here will be $\pm\left(\frac{1}{21}\right) \cdot 4 \approx \pm 0.19$.

2. $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left(\frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}}\right)$

$$= \arctan 1$$
$$= \pi/4.$$

2' so $\pi = 4\left(\arctan\frac{1}{3} + \arctan\frac{1}{2}\right) = 4 \sum (-1)^n \frac{1}{2n+1}\left[\frac{1}{3^{2n+1}} + \frac{1}{2^{2n+1}}\right]$

To approximate π to 2 decimal points, need margin of error $< \frac{1}{100} = \frac{4}{2n+1}\left[\frac{1}{3^{2n+1}} + \frac{1}{2^{2n+1}}\right]$, use $n = 3$

$$\pi = 4 \sum_{n=0}^{3} (-1)^n \frac{1}{2n+1}\left[\frac{1}{3^{2n+1}} + \frac{1}{2^{2n+1}}\right]$$
$$= 4\left[\frac{1}{3} + \frac{1}{2} - \frac{1}{3\cdot 3^3} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 3^5} + \frac{1}{5\cdot 2^5}\right]$$
$$= 3.14 \text{ with margin of error } \pm 4\left(\frac{1}{7\cdot 3^7} + \frac{1}{7\cdot 2^7}\right) = \pm 0.00472.$$

3. By Machin's formula:

$$\pi = 16 \arctan \frac{1}{5} - 4 \arctan \frac{1}{239}$$

And by using Taylor's expansion:

$$\pi = 16\left(\frac{1}{5} - \frac{1}{3}\cdot\left(\frac{1}{5}\right)^3 + \frac{1}{5}\cdot\left(\frac{1}{5}\right)^5 - \dots\right)$$

$$- 4\left(\frac{1}{239} - \frac{1}{3}\cdot\left(\frac{1}{239}\right)^3 + \frac{1}{5}\cdot\left(\frac{1}{239}\right)^5 - \dots\right)$$

To estimate up to 3 decimal points, need margin of error $< \frac{1}{1000}$. In this case, we can see that using just 3 terms will give a margin error $< \frac{16}{7}\cdot\left(\frac{1}{5}\right)^7 - \frac{4}{7}\left(\frac{1}{239}\right)^7 < \frac{1}{1000}$

So:

$$\pi \approx 16\left(\frac{1}{5} - \frac{1}{3}\cdot\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{1}{5}\right)^5\right)$$

$$- 4\left(\frac{1}{239} - \frac{1}{3}\cdot\left(\frac{1}{239}\right)^3 + \frac{1}{5}\left(\frac{1}{239}\right)^5\right)$$

$$\underline{\pi \approx 3.141 \pm 0.00002}$$

4. $$\frac{128}{\pi^2} = 13 - \left(\frac{1}{2}\right)^5 \frac{20\cdot 50 + 13}{2^{10}} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^5 \frac{40\cdot 91 + 13}{2^{20}} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^5 \frac{60\cdot 132 + 13}{2^{30}}$$

$$= 12.96911$$

then $\pi = \sqrt{\frac{128}{12.96911\dots}}$

$$\pi = \underbrace{3.14159265359}\dots$$

This result corresponds to 11 digits of the actual π value.