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10/24/21

Homework for Lecture 12 - OK to post

① We use the formula $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ to find the first three terms $(n=0,1,2)$

Since $f(x) = \sin(x+x^2)$, we compute

$f'(x) = (1+2x)\cos(x+x^2)$

$f''(x) = (1+2x)[(1+2x)(-\sin(x+x^2))] + [\cos(x+x^2)]2$
$= -(1+2x)^2 \sin(x+x^2) + 2\cos(x+x^2)$

Substituting in $x=0$, we get

$f(0) = 0$
$f'(0) = 1$
$f''(0) = 2$

Thus the first three terms are $0 + x + x^2$, which is equivalent to $x + x^2$

② We use the expansion for $\sin(z)$, which is

$$\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \cdots$$

Next, we plug in $(x+x^2)$ for z, to obtain

$$(x+x^2) - \frac{1}{3!}(x+x^2)^3 + \frac{1}{5!}(x+x^2)^5 - \frac{1}{7!}(x+x^2)^7$$

We expand and simplify, while discarding any terms with degree 5 or higher

$$x + x^2 - \frac{1}{6}x^3 - \frac{1}{2}x^4$$

where the first term is 0, as shown in problem 1, so these are the first five terms ($n = 0, 1, 2, 3, 4$), which is up to and including the fourth power.

③ We use the fact that

$$\arctan x = \int_0^x \frac{1}{1+t^2}$$

and that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

We make the replacement $z = -t^2$ to get that

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-t^2)^n = \sum_{n=0}^{\infty} (-1)^n t^{2n}$$

Thus

$$\arctan x = \int_0^x \frac{1}{1+t^2} = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n} = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n} = \sum_{n=0}^{\infty} (-1)^n \left[\frac{t^{2n+1}}{2n+1}\right]_0^x$$

$$= \underbrace{\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}}_{\text{Taylor series around } x=0 \text{ of } \arctan x}$$

Taylor series around $x = 0$ of $\arctan x$

(4) Let $a = \arctan x$ and $b = \arctan y$

Therefore, $\tan(a) = x$
$\tan(b) = y$

We set up the equation

$$\tan(\arctan(x) + \arctan(y)) = \tan(a+b)$$

We can rewrite the right hand side by using the trig identity

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - (\tan\alpha)(\tan\beta)} \quad \text{where } \alpha = a \text{ and } \beta = b$$

Thus

$$\tan(\arctan(x) + \arctan(y)) = \frac{\tan a + \tan b}{1 - (\tan a)(\tan b)}$$

Since $\tan(a) = x$ and $\tan(b) = y$, we can rewrite this as

$$\tan(\arctan(x) + \arctan(y)) = \frac{x+y}{1-xy}$$

Taking the arctan of both sides, we obtain

$$\arctan(x) + \arctan(y) = \arctan\frac{x+y}{1-xy}$$

(5) Using the formula derived in the previous problem, we plug in $x = \frac{1}{2}$ and $y = \frac{1}{3}$

$$\arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right) = \arctan\frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}$$

$$= \arctan\frac{\frac{5}{6}}{\frac{5}{6}}$$

$$= \arctan\left(\frac{\pi}{4}\right)$$

$$= 1$$