Sarah Catovic (ok to post)   Homework 12   10/24/2021

1. $f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$ $f(x) = \sin(x + x^2)$

$f'(x) = (1+2x)\cos(x+x^2)$ $f''(x) = 2\cos(x+x^2) + (1+2x)^2(-\sin(x+x^2))$

First 3 terms: $\frac{\sin(0)}{0!} x^0 = 0(1) = 0$

$\frac{\cos(0)}{1!} x^1 = \frac{1}{1!} x^1 = x$

$\frac{2\cos(0) + (1)^2(-\sin(0))}{2!} x^2 = \frac{2}{2!} x^2 = x^2$

2. $\sin z = z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \frac{1}{7!} z^7$ $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$\sin(x+x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x+x^2)^{2n+1}}{(2n+1)!}$

First 5 terms: $(x+x^2) - \frac{(x+x^2)^3}{3!} + \frac{(x+x^2)^5}{5!} - \frac{(x+x^2)^7}{7!} + \frac{(x+x^2)^9}{9!}$

3. $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ Let $z = -x^2$

$\frac{1}{1-(-x^2)} = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$\arctan(x) = \int \frac{1}{1+x^2} = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}$

$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

So $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

4. $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - (\tan\alpha)(\tan\beta)}$ $x = \tan\alpha$ $y = \tan\beta$

$\alpha = \arctan(x)$ $\beta = \arctan(y)$

$\arctan(\tan(\alpha+\beta)) = \arctan\left(\frac{\tan\alpha + \tan\beta}{1-(\tan\alpha)(\tan\beta)}\right)$

$\alpha + \beta = \arctan\left(\frac{x+y}{1-xy}\right)$

$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right)$

5. Using the previously derived identity: $x = \frac{1}{2}$ $y = \frac{1}{3}$

$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}$$

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}\right)$$

$$\downarrow$$

$$\arctan\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right) = \arctan(1) = \pi/4$$

so $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan(1) = \pi/4$