Anna Zhebrun
OK to post

Homework 12 10/23/2021

1. Taylor expansion of $f(x) = \sin(x + x^2)$ about $x = 0$

$f^0(0) = \sin(0 + 0^2) = 0$

$f'(0) = (1 + 2\cdot 0)\cos(0 + 0^2) = 1$ $\quad f^2(x) = (1+2x)\cos(x+x^2)$

$f^2(0) = 2$ $\quad f^3(x) = (1+2x)^2 \sin(x+x^2) + 2\cos(x+x^2)$

$f^3(0) = -1$ $\quad f^4(x) = -(1+2x)^3 \cos(x+x^2) - 6(1+2x)\sin(x+x^2)$

Then, Taylor Expansion becomes:

$$f(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f^2(0)}{2!}(x-0)^2$$

$$\boxed{f(x) = 0 + x + x^2}$$ for the first 3 terms.

2. Since $\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \frac{1}{9!}z^9 + \dots$ and we want to find Taylor expansion for the first five terms of $\sin(x+x^2)$, we are interested in the powers up to x^4

$$\sin(x+x^2) = x(x+1) - \frac{1}{3!}x^3(x+1)^3 + \frac{1}{5!}x^5(x+1)^5$$

$$= x^2 + x - \frac{1}{3!}(x^6 + 3x^5 + 3x^4 + x^3) + \dots$$

$$= \underbrace{x + x^2 - \frac{1}{6}x^3 - \frac{1}{2}x^4}_{} + \dots$$

(Do not calculate past $\frac{1}{3!}z$, because all powers after will be of bigger degree than 4).

Scanned with CamScanner

3. Since $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, make the substitution $z = -t^2$. Then: $\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n (t^2)^n$. Now calculate:

$$\arctan x = \int_0^x \frac{1}{1+t^2}\, dt$$

$$= \int_0^x \sum_{n=0}^{\infty} (-1)^n (t^2)^n \, dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\, dt = \sum_{n=0}^{\infty} (-1)^n \left(\frac{t^{2n+1}}{2n+1}\right)\Big|_0^x$$

$$\boxed{\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

4. take $\tan \alpha = x$ and $\tan \beta = y$.

then $\arctan x = \alpha$ and $\arctan y = \beta$

Start with:

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}.$$

Take arctan of both sides:

$$\arctan(\tan(\alpha+\beta)) = \arctan\left(\frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}\right)$$

$$\alpha + \beta = \arctan\left(\frac{x+y}{1-xy}\right)$$

since $\tan\alpha = x$ and $\tan\beta = y$ and $\arctan(\tan z) = z$. So:

$$\boxed{\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)}$$

Because $\alpha = \arctan x$ and $\beta = \arctan y$.

Scanned with CamScanner

5. Prove $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$.

Using the arctan identity:

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)$$

$$= \arctan\left(\frac{\frac{1}{6}}{1-\frac{5}{6}}\right)$$

$$= \arctan(1).$$

$$= \pi/4, \text{ (because } \tan\frac{\pi}{4} = 1)$$

Therefore:

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}.$$

//