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Homework 12

(1) $f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$ to find the first 3 terms (i.e. $n = 0, 1, 2$ of the Taylor Expansion around $x = 0$ of $f(x) = \sin(x+x^2)$.

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2$$

$$f(x) = 0 + \frac{1}{1!} x + \frac{2}{2!} x^2$$

$$f(x) = 0 + x + x^2$$

(2) Using the known formula $\sin z = z - 1/3!\,(z^3) + 1/5!\,(z^5) - 1/7!\,(z^7) + \cdots$ and high school algebra (pretending the power series is like polynomials), to find the first 5 terms (i.e. $n = 0, 1, 2, 3, 4$) of the Taylor Expansion around $x = 0$ of $f(x) = \sin(x+x^2)$.

$$\sin(z) = z - \frac{1}{3!}(z^3) + \frac{1}{5!}(z^5) + \cdots$$

$$\sin(x+x^2) = (x+x^2) - \frac{1}{3!}(x+x^2)^3 + \frac{1}{5!}(x+x^2)^5 - \frac{1}{7!}(x-x^2)^7 + \frac{1}{9!}(x+x^2)^9$$

$$\sin(x+x^2) = x+x^2 - \frac{1}{3!}(x^6+3x^5+3x^4+x^3) - \frac{1}{7!}(x-x^2)^7 + \frac{1}{9!}(x+x^2)^9$$

$$\sin(x+x^2) = x+x^2 - \frac{x^3}{6} - \frac{x^4}{2}$$

(3) Using the fact that $\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$ and the famous Taylor series for $\frac{1}{1-x}$ $\frac{1}{1-z} = \sum_{n=0}^{\infty} x^n$, derive the taylor series (around $x = 0$) of the function $\arctan x$.

By replacing z with $-t^2$, we get

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-t^2)^n = \sum_{n=0}^{\infty} (-1)^n t^{2n}$$

Hence $\arctan x = \int_0^x \frac{1}{1+t^2} = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n} = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \left(\frac{t^{2n+1}}{2n+1} \Bigg|_0^x \right) = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

(4) Using the trig identity $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan(\alpha)\tan(\beta)}$ derive the arctan identity

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{(x+y)}{1-xy}\right)$$

Let $\arctan x = \alpha$ and $\arctan y = \beta$.

$x = \tan(\alpha)$ and $y = \tan(\beta)$.

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan(\alpha)\tan(\beta)}.$$

$$\alpha+\beta = \arctan\left(\frac{\tan\alpha + \tan\beta}{1-\tan(\alpha)\tan(\beta)}\right)$$

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right)$$

(5) Prove that $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$

$$\arctan(x) + \arctan(y) = \arctan\left(\frac{x+y}{1-xy}\right)$$

$$\arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right) = \arctan\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}\right)$$

$$\arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right) = \arctan\left(\frac{\frac{3}{6}+\frac{2}{6}}{1-\left(\frac{1}{6}\right)}\right)$$

$$\arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right) = \arctan\left(\frac{5/6}{5/6}\right)$$

$$\arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right) = \arctan(1)$$

$$\arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right) = \pi/4.$$